Homework #8
CS675 Fall 2023

Due Monday Nov 13, by midnight

General Instructions The following assignment is meant to be challenging. Feel free to discuss with fellow students, though please write up your solutions independently and acknowledge everyone you discussed the homework with on your writeup. I also expect that you will not attempt to consult outside sources, on the Internet or otherwise, for solutions to any of these homework problems — doing so would be considered cheating.

Several of these problems are drawn from the following texts, each of which is linked on the course website: Luenberger and Ye (4th edition), Korte and Vygen (5th edition), and Boyd and Vendenberghe. Please make sure you are using the correct edition of each of the books by using the links on the course website.

We request that you submit your homework as a pdf file, by email to the TA.

Finally, whenever a question asks you to "show" or "prove" a claim, please provide a formal mathematical proof.

Problem 1. (20 points)
In this problem, we will examine some miscellaneous properties of matroids. We use $\mathcal{M} = (\mathcal{X}, \mathcal{I})$ to denote an arbitrary matroid.

(a) [4 points]. Prove that the rank function $rank_{\mathcal{M}} : 2^{\mathcal{X}} \rightarrow \mathbb{N}$ of a matroid is monotone, normalized, and submodular.

(b) [4 points]. Prove the *strong exchange property* of matroids: If B and B' are bases of matroid $\mathcal{M}$, then for every $i \in B \setminus B'$ there is some $j \in B' \setminus B$ such that $(B \setminus \{i\}) \cup \{j\}$ is a basis of $\mathcal{M}$. Show that the exchange property is implied by the strong exchange property for every downwards-closed set system. (i.e. we could have equivalently defined matroids using the strong exchange property instead).

(c) [4 points]. Show that matroids are closed under *truncation*. Namely, if $\mathcal{M} = (\mathcal{X}, \mathcal{I})$ is a matroid, and k is a nonnegative integer, then the set system $\mathcal{M}_k = (\mathcal{X}, \mathcal{I}_k)$ with $\mathcal{I}_k = \{S \in \mathcal{I} : |S| \leq k\}$ is also a matroid.

(d) [4 points]. Recall that we call a set $C \subseteq \mathcal{X}$ a *circuit* of the matroid $\mathcal{M}$ if it is a minimal dependent set in $\mathcal{M}$. Note that a circuits of a graphical matroid correspond to cycles in the graph. Prove the *circuit property* of matroids: if $\mathcal{M} = (\mathcal{X}, \mathcal{I})$ is a matroid with distinct weights $w \in \mathbb{R}^{\mathcal{X}}$

assigned to its ground set, and C is a circuit of $\mathcal{M}$, then the element $i \in C$ of minimum weight is not in any maximum-weight basis of $\mathcal{M}$.

(e) [4 points]. We call a set $C \subseteq \mathcal{X}$ a *cut* of the matroid $\mathcal{M}$ if every basis of $\mathcal{M}$ intersects C, and no proper subset of C has this property. Note that a cut of a graphical matroid correspond to our notion of a cut in a graph — namely, a family of edges who's removal disconnects some connected component of the graph. Prove the *cut property* of matroids: if $\mathcal{M} = (\mathcal{X}, \mathcal{I})$ is a matroid with distinct weights $w \in \mathbb{R}^{\mathcal{X}}$ assigned to its ground set, and C is a cut of $\mathcal{M}$, then the element $i \in C$ of maximum weight is in every maximum-weight basis of $\mathcal{M}$.

Problem 2. (10 points)
In class, we showed that a simple greedy algorithm maximizes a modular function over the independent sets of a matroid. Consider instead the problem of minimizing a modular function over the bases of a matroid. Show that a similar greedy algorithm succeeds for this problem.
Note: This is a generalization of the minimum spanning tree problem.
Hint: You can either mimic the proof presented in class for maximizing a modular function over the independent sets of a matroid, or for a shorter proof you can use a reduction to that problem!

Problem 3. (15 points)
Given two matroids $\mathcal{M}_1 = (\mathcal{X}, \mathcal{I}_1)$ and $\mathcal{M}_2 = (\mathcal{X}, \mathcal{I}_2)$ with a common ground set $\mathcal{X}$, and a weight vector $w \in \mathbb{R}^{\mathcal{X}}$, the *maximum-weight common independent set* of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the set $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ maximizing $\sum_{i \in S} w(i)$. We discussed in class how the maximum-weight common independent set can be computed in time $\text{poly}(|\mathcal{X}|)$, given only independence-oracle access to the matroids $\mathcal{M}_1$ and $\mathcal{M}_2$. Recall that this followed from the integrality of the polytope $\mathcal{P}(\mathcal{M}_1) \cap \mathcal{P}(\mathcal{M}_2)$, where $\mathcal{P}(\mathcal{M}_1)$ and $\mathcal{P}(\mathcal{M}_2)$ are the two matroid polytopes.

We will now consider a related problem, the *minimum-weight common-basis* problem, show how to solve it efficiently, and show that it can be viewed as a generalization of the shortest path problem in directed graphs. Given two matroids $\mathcal{M}_1 = (\mathcal{X}, \mathcal{I}_1)$ and $\mathcal{M}_2 = (\mathcal{X}, \mathcal{I}_2)$, let $\mathcal{B}_1 \subseteq 2^{\mathcal{X}}$ denote the family of all bases (maximum-cardinality independent sets) of $\mathcal{M}_1$, and similarly $\mathcal{B}_2$ for $\mathcal{M}_2$. Given a weight vector $w \in \mathbb{R}^{\mathcal{X}}$, the minimum-weight common basis of $\mathcal{M}_1$ and $\mathcal{M}_2$, if it exists, is the set $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ minimizing $\sum_{i \in B} w(i)$. Note that for a common basis to exist, both matroids must have the same rank (though this is not sufficient).

(a) [8 points]. Show that the minimum-weight common-basis problem can be solved in polynomial time in $|\mathcal{X}|$, given only independence-oracle access to the two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$. Your algorithm should either output a minimum-weight common-basis of $\mathcal{M}_1$ and $\mathcal{M}_2$ if one exists, or else assert that there is no common basis.
(**Hint**: You can start by showing that that the convex hull of common bases is a face of the matroid intersection polytope)

(b) [7 points]. Show that the shortest path problem in weighted directed graphs can be reduced, in linear time, to the minimum-weight common-basis problem when the graph has no negative cycles.
(**Hint**: Add self-loops to all nodes other than the source and destination, and define two partition matroids such that their common bases are precisely a path combined with some of the self-loops.)

Problem 4. (10 points)
Let $\mathcal{M} = (\mathcal{X}, \mathcal{I})$ be a matroid. The *matroid covering theorem* states that the elements can be partitioned into k independent sets if and only if $|S| \leq k \cdot rank_{\mathcal{M}}(S)$ for every $S \subseteq \mathcal{X}$. I won't ask you to prove this theorem, but we will instead examine its algorithmic implications.

(a) [5 points]. Show how to test, in polynomial time in the independence oracle model, whether a matroid can be partitioned into k independent sets.

(b) [5 points]. Given an undirected graph G, we wish to color the edges so that no cycle is monochromatic (i.e., the edges of the cycle do not all have the same color). Describe a polynomial time algorithm to determine the minimum number of distinct colors needed.