CS675: Convex and Combinatorial Optimization
Fall 2023
Convex Optimization Problems

Instructor: Shaddin Dughmi

Outline

Recall: Convex Optimization Problem

A problem of minimizing a convex function (or maximizing a concave function) over a convex set.

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

- $\mathcal{X} \subseteq \mathbb{R}^n$ is convex, and $f : \mathbb{R}^n \to \mathbb{R}$ is convex
- Terminology: decision variable(s), objective function, feasible set, optimal solution/value, ϵ-optimal solution/value

Standard Form

Instances typically formulated in the following standard form

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \leq 0, & \text{for } i \in \mathcal{C}_1. \\ & \langle a_i, x \rangle = b_i, & \text{for } i \in \mathcal{C}_2. \end{array}$$

- g_i is convex
- Terminology: equality constraints, inequality constraints, active/inactive at x, feasible/infeasible, unbounded

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  - Recall: every convex set is the intersection of halfspaces

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- In principle, every convex optimization problem can be formulated in this form (possibly implicitly)
  - Recall: every convex set is the intersection of halfspaces
- When there is no objective function (or, equivalently, $f(x) = 0$ for all x), we say this is convex feasibility problem

Local and Global Optimality

$x \in \mathcal{X}$ is locally optimal if $\exists$ open ball B centered at x s.t. $f(x) \leq f(y)$ for all $y \in B \bigcap \mathcal{X}$. It is globally optimal if it's an optimal solution.

Fact

For a convex optimization problem, every locally optimal feasible solution is globally optimal.

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- Let x be locally optimal, and y be any other feasible point.
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- By local optimality $f(x) \leq f(\theta x + (1-\theta)y)$ for θ sufficiently close to 1.

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- The line segment from x to y is contained in the feasible set.
- By local optimality $f(x) \leq f(\theta x + (1-\theta)y)$ for θ sufficiently close to 1.
- Jensen's inequality then implies that y is suboptimal.

$$f(x) \leq f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

$$f(x) \leq f(y)$$

Representation

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Explicit Representation

A family of linear programs of the following form

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x \succeq 0 \end{array}$$

may be described by $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$.

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Oracle Representation

At their most abstract, convex optimization problems of the following form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

are described via a separation oracle for $\mathcal{X}$ and $\mathbf{epi}\, f$.

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Given additional data about instances of the problem, namely a range $[L, H]$ for its optimal value and a ball of volume V containing $\mathcal{X}$, the ellipsoid method returns an ϵ-optimal solution using only $\mathrm{poly}(n, \log(\frac{H-L}{\epsilon}), \log V)$ oracle calls.

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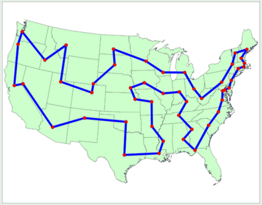
In Between

Consider the following fractional relaxation of the Traveling Salesman Problem, described by a network (V, E) and distances d_e on $e \in E$.



min $\sum_e d_e x_e$
s.t.
$\sum_{e \in \delta(S)} x_e \geq 2, \quad \forall S \subset V, S \neq \emptyset.$
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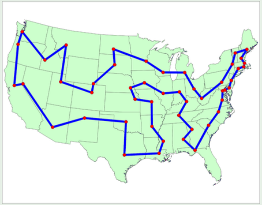

Representation of LP is implicit, in the form of a network. Using this representation, separation oracles can be implemented efficiently, and used as subroutines in the ellipsoid method.

Equivalence

- Next up: we look at some common classes of convex optimization problems
- Technically, not all of them will be convex in their natural representation
- However, we will show that they are “equivalent” to a convex optimization problem

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Note

Deciding whether an optimization problem is equivalent to a tractable convex optimization problem is, in general, a black art honed by experience. There is no silver bullet.

Outline

Linear Programming

We have already seen linear programming

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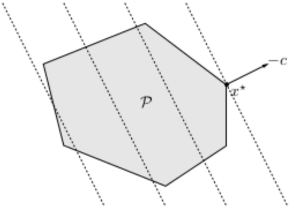

Linear Fractional Programming

Generalizes linear programming

$$
\begin{array}{ll}
\text{minimize} & \frac{\langle c,x\rangle + d}{\langle e,x\rangle + f} \\
\text{subject to} & Ax \preceq b \\
& \langle e,x\rangle + f > 0
\end{array}
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- The objective is quasiconvex (in fact, quasilinear) over the open halfspace where the denominator is positive.

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- Can be reformulated as an equivalent linear program
  1. Change variables to $y = \frac{x}{\langle e,x\rangle + f}$ and $z = \frac{1}{\langle e,x\rangle + f}$

$$\begin{array}{ll} \text{minimize} & \langle c,y\rangle + dz \\ \text{subject to} & Ay \preceq bz \\ & z > 0 \\ & y = \frac{x}{\langle e,x\rangle + f} \\ & z = \frac{1}{\langle e,x\rangle + f} \end{array}$$

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Example: Optimal Production Variant

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j dollars per unit, and requires an investment of e_j dollars per unit to produce, with f as a fixed cost
- Facility wants to maximize "Return rate on investment"

$$
\begin{array}{lll}
\text{maximize} & \frac{\langle c,x\rangle}{\langle e,x\rangle+f} & \\
\text{subject to} & \langle a_i, x\rangle \leq b_i, & \text{for } i = 1, \ldots, m. \\
 & x_j \geq 0, & \text{for } j = 1, \ldots, n.
\end{array}
$$

Geometric Programming

Definition

- A monomial is a function $f : \mathbb{R}_+^n \to \mathbb{R}_+$ of the form

$$f(x) = cx_1^{a_1} x_2^{a_2} \ldots x_n^{a_n},$$

  where $c \geq 0$, $a_i \in \mathbb{R}$.
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Interpretation

GP model volume/area minimization problems, subject to constraints.

Example: Designing a Suitcase

- A suitcase manufacturer is designing a suitcase
- Variables: h, w,d
- Want to minimize surface area $2(hw + hd + wd)$ (i.e. amount of material used)
- Have a target volume $hwd \geq 5$
- Practical/aesthetic constraints limit aspect ratio: $h/w \leq 2$, $h/d \leq 3$
- Constrained by airline to $h + w + d \leq 7$

$$
\begin{array}{ll}
\text{minimize} & 2hw + 2hd + 2wd \\
\text{subject to} & h^{-1}w^{-1}d^{-1} \leq \frac{1}{5} \\
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More interesting applications involve optimal component layout in chip design.

Designing a Suitcase in Convex Form

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Change of variables to $\widetilde{h} = \log h$, $\widetilde{w} = \log w$, $\widetilde{d} = \log d$

$$
\begin{array}{ll}
\text{minimize} & 2e^{\widetilde{h}+\widetilde{w}} + 2e^{\widetilde{h}+\widetilde{d}} + 2e^{\widetilde{w}+\widetilde{d}} \\
\text{subject to} & e^{-\widetilde{h}-\widetilde{w}-\widetilde{d}} \leq \frac{1}{5} \\
& e^{\widetilde{h}-\widetilde{w}} \leq 2 \\
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& e^{\widetilde{h}} + e^{\widetilde{w}} + e^{\widetilde{d}} \leq 7
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Geometric Programs in Convex Form

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where f_i's are posynomials, h_i's are monomials, and $b_i > 0$ (wlog 1).

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- However, the feasible set and objective function are convex in the variables $y_1, \ldots, y_n \in \mathbb{R}$ where $y_i = \log x_i$

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- Each monomial $cx_1^{a_1}x_2^{a_2}\ldots x_k^{a_k}$ can be rewritten as a convex function $ce^{a_1y_1+a_2y_2+\ldots+a_ky_k}$
- Therefore, each posynomial becomes the sum of these convex exponential functions
- Inequality constraints and objective become convex
- Equality constraint $cx_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} = b$ reduces to an affine constraint $a_1y_1 + a_2y_2 \ldots a_ky_k = \log\frac{b}{c}$

Outline

Symmetric Matrices

A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if and only if it is square and $A_{ij} = A_{ji}$ for all i, j.

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Fact

A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if and only if it is orthogonally diagonalizable.

- i.e. $A = QDQ^\intercal$ where Q is an orthogonal matrix and $D = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$.
- The columns of Q are the (normalized) eigenvectors of A, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$
- Equivalently: As a linear operator, A scales the space along an orthonormal basis Q
- The scaling factor λ_i along direction q_i may be negative, positive, or 0.

Positive Semi-Definite Matrices

A matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite if it is symmetric and moreover all its eigenvalues are nonnegative.

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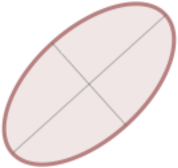
Note

Positive definite, negative semi-definite, and negative definite defined similarly.

Geometric Intuition for PSD Matrices



- For $A \succeq 0$, let $q_1, \ldots, q_n$ be the orthonormal eigenbasis for A, and let $\lambda_1, \ldots, \lambda_n \geq 0$ be the corresponding eigenvalues.
- The linear operator $x \to Ax$ scales the q_i component of x by λ_i
- When applied to every x in the unit ball, the image of A is an ellipsoid centered at the origin with principal directions $q_1, \ldots, q_n$ and corresponding diameters $2\lambda_1, \ldots, 2\lambda_n$
  - When A is positive definite ($i.e. \lambda_i > 0$), and therefore invertible, the ellipsoid is the set $\left\{y : y^T (AA^T)^{-1} y \leq 1\right\}$

Useful Properties of PSD Matrices

If $A \succeq 0$, then

- $x^T A x \geq 0$ for all x
- A has a positive semi-definite square root $A^{\frac{1}{2}}$
  - $A^{\frac{1}{2}} = Q\,\mathbf{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})Q^{\intercal}$
- $A = B^T B$ for some matrix B.
  - Interpretation: PSD matrices encode the “pairwise similarity” relationships of a family of vectors. A_{ij} is dot product of the ith and jth columns of B.
  - Interpretation: The quadratic form $x^T A x$ is the length of a linear transformation of x, namely $||Bx||_2^2$
- The quadratic function $x^T A x$ is convex
- A can be expressed as a sum of vector outer-products
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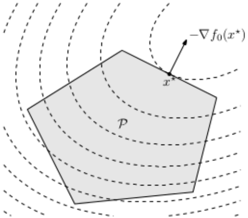
- A can be expressed as a sum of vector outer-products
  - e.g., $A = \sum_{i=1}^n v_i v_i^T$ for $\vec{v_i} = \sqrt{\lambda_i}\vec{q_i}$

As it turns out, each of the above is also sufficient for $A \succeq 0$ (assuming A is symmetric).

Outline

Minimizing convex quadratic fn over a polyhedron. Require $P \succeq 0$.

$$\begin{array}{ll} \text{minimize} & x^\intercal P x + \langle c, x \rangle + d \\ \text{subject to} & Ax \leq b \end{array}$$



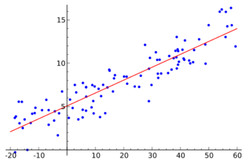
- When $P \succ 0$, objective can be rewritten as $(x - x_0)^\intercal P (x - x_0)$ for some center x_0 (might need to change d, which is immaterial)
    - Sublevel sets are scaled copies of an ellipsoid centered at x_0

Examples

Constrained Least Squares

Given a set of measurements $(a_1, b_1), \ldots, (a_m, b_m)$, where $a_i \in \mathbb{R}^n$ is the i'th input and $b_i \in \mathbb{R}$ is the i'th output, fit a linear function minimizing mean square error, subject to known bounds on the linear coefficients.

$$\begin{array}{lll} \text{minimize} & ||Ax - b||_2^2 = & x^\intercal A^\intercal Ax - 2b^\intercal Ax + \langle b, b \rangle \\ \text{subject to} & l_i \leq x_i \leq u_i, & \text{for } i = 1, \ldots, n. \end{array}$$



Examples

Distance Between Polyhedra

Given two polyhedra $Ax \preceq b$ and $Cx \preceq d$, find the distance between them.

$$\begin{array}{ll} \text{minimize} & ||z||_2^2 = z^\mathsf{T} I z \\ \text{subject to} & z = y - x \\ & Ax \preceq b \\ & By \preceq d \end{array}$$

Conic Optimization Problems

This is an umbrella term for problems of the following form

$$\begin{array}{ll} \text{minimize} & c^\intercal x \\ \text{subject to} & Ax + b \in K \end{array}$$

Where K is a convex cone (e.g. $\mathbb{R}^n_+$, positive semi-definite matrices, etc). Evidently, such optimization problems are convex.

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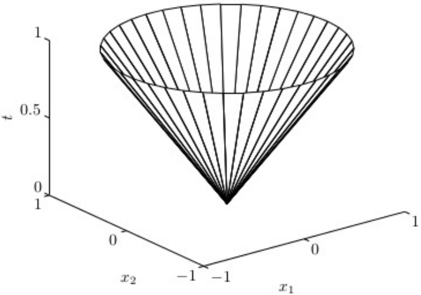
As shorthand, the cone containment constraint is often written using generalized inequalities

- $Ax + b \succeq_K 0$
- $-Ax \preceq_K b$
- $\ldots$

Example: Second Order Cone Programming

We will exhibit an example of a conic optimization problem with K as the second order cone

$$K = \{(x, t) : ||x||_2 \leq t\}$$



Example: Second Order Cone Programming

Linear Program with Random Constraints

Consider the following optimization problem, where each a_i is a gaussian random variable with mean $\overline{a}_i$ and covariance matrix Σ_i.

$$\begin{array}{ll} \text{minimize} & c^\intercal x \\ \text{subject to} & \langle a_i, x \rangle \leq b_i \textbf{ w.p. } \text{at least } 0.9, \quad \text{for } i = 1, \ldots, m. \end{array}$$

- $u_i := \langle a_i, x \rangle$ is a univariate normal r.v. with mean $\overline{u}_i := \overline{a}_i^\intercal x$ and stddev $\sigma_i := \sqrt{x^\intercal \Sigma_i x} = ||\Sigma_i^{\frac{1}{2}} x||_2$

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- $u_i \leq b_i$ with probability $\phi(\frac{b_i - \overline{u}_i}{\sigma_i})$, where ϕ is the CDF of the standard normal random variable.
- Since we want this probability to exceed 0.9, we require that

$$\frac{b_i - \overline{u}_i}{\sigma_i} \geq \phi^{-1}(0.9) \approx 1.3 \approx 1/0.77$$

$$||\Sigma_i^{\frac{1}{2}} x||_2 \leq 0.77(b_i - \overline{a}_i^\intercal x)$$

Semi-Definite Programming

These are conic optimization problems where the cone in question is the set of positive semi-definite matrices.

$$\begin{array}{ll} \text{minimize} & c^\intercal x \\ \text{subject to} & x_1 F_1 + x_2 F_2 \ldots x_n F_n + G \succeq 0 \end{array}$$

Where $F_1, \ldots, F_n$ are matrices, and $\succeq$ refers to the positive semi-definite cone S_+^n.

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Examples

- Fitting a distribution, say a Gaussian, to observed data. Variable is a positive semi-definite covariance matrix.
- As a relaxation to combinatorial problems that encode pairwise relationships: e.g. finding the maximum cut of a graph.

Example: Max Cut Problem

Given an undirected graph $G = (V, E)$, find a partition of V into $(S, V \setminus S)$ maximizing number of edges with exactly one end in S.

$$\begin{array}{ll} \text{maximize} & \sum_{(i,j)\in E} \frac{1-x_i x_j}{2} \\ \text{subject to} & x_i \in \{-1, 1\}\,, \qquad \text{for } i \in V. \end{array}$$

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Vector Program relaxation

$$\begin{array}{lll} \text{maximize} & \sum_{(i,j)\in E} \frac{1-x_i \cdot x_j}{2} & \\ \text{subject to} & ||x_i||_2 = 1, & \text{for } i \in V. \\ & x_i \in \mathbb{R}^n, & \text{for } i \in V. \end{array}$$

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SDP Relaxation

$$\begin{array}{lll} \text{maximize} & \sum_{(i,j)\in E} \frac{1 - X_{ij}}{2} & \\ \text{subject to} & X_{ii} = 1, & \text{for } i \in V. \\ & X \in S^n_+ & \end{array}$$