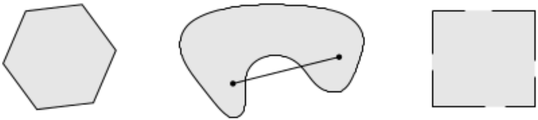
CS675: Convex and Combinatorial Optimization
Fall 2023
Convex Sets

Instructor: Shaddin Dughmi

Outline

Convex Sets

A set $S \subseteq \mathbb{R}^n$ is convex if the line segment between any two points in S lies in S. i.e. if $x, y \in S$ and $\theta \in [0,1]$, then $\theta x + (1-\theta)y \in S$.



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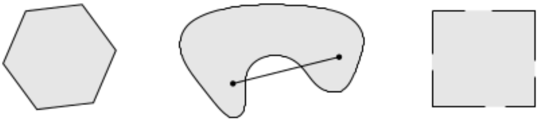


Equivalent Definition

S is convex if every convex combination of points in S lies in S.

Convex Combination

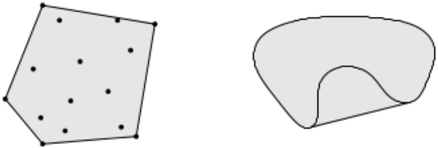
- Finite: y is a convex combination of $x_1, \ldots, x_k$ if $y = \theta_1 x_1 + \ldots \theta_k x_k$, where $\theta_i \geq 0$ and $\sum_i \theta_i = 1$.
- General: expectation of probability measure on S.

Convex Sets

Convex Hull

The convex hull of $S \subseteq \mathbb{R}^n$ is the smallest convex set containing S.

- Intersection of all convex sets containing S
- The set of all convex combinations of points in S



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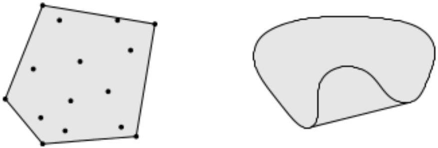
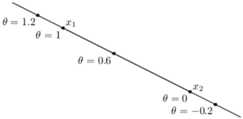


A set S is convex if and only if convexhull$(S) = S$.

Affine Set

A set $S \subseteq \mathbb{R}^n$ is affine if the line passing through any two points in S lies in S. i.e. if $x, y \in S$ and $\theta \in \mathbb{R}$, then $\theta x + (1 - \theta)y \in S$.



Obviously, affine sets are convex.

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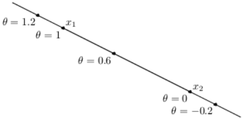

Obviously, affine sets are convex.

Equivalent Definition

S is affine if every affine combination of points in S lies in S.

Affine Combination

y is an affine combination of $x_1, \ldots, x_k$ if $y = \theta_1 x_1 + \ldots \theta_k x_k$, and $\sum_i \theta_i = 1$.

Generalizes convex combinations

Affine Sets

Equivalent Definition II

S is affine if and only if it is a shifted subspace

- i.e. $S = x_0 + V$, where V is a linear subspace of $\mathbb{R}^n$.

- Any $x_0 \in S$ will do, and yields the same V.
- The dimension of S is the dimension of subspace V.

Affine Sets

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Equivalent Definition III

S is affine if and only if it is the solution of a set of linear equations (i.e. the intersection of hyperplanes).

- i.e. $S = \{x : Ax = b\}$ for some matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Affine Sets

Affine Hull

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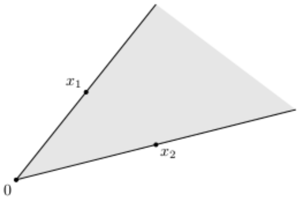
Affine Dimension

The affine dimension of a set is the dimension of its affine hull

Cones

A set $K \subseteq \mathbb{R}^n$ is a cone if the ray from the origin through every point in K is in K i.e. if $x \in K$ and $\theta \geq 0$, then $\theta x \in K$.



Note: every cone contains 0.

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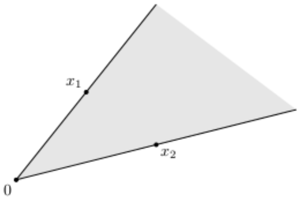


Note: every cone contains 0.

Special Cones

- A convex cone is a cone that is convex
- A cone is pointed if whenever $x \in K$ and $x \neq 0$, then $-x \notin K$.
- We will mostly mention proper cones: convex, pointed, closed, and of full affine dimension.

Cones

Equivalent Definition

K is a convex cone if every conic combination of points in K lies in K.

Conic Combination

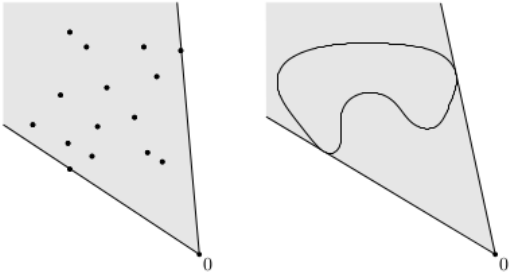
y is a conic combination of $x_1, \ldots, x_k$ if $y = \theta_1 x_1 + \ldots \theta_k x_k$, where $\theta_i \geq 0$.

Cones

Conic Hull

The conic hull of $K \subseteq \mathbb{R}^n$ is the smallest convex cone containing K

- Intersection of all convex cones containing K
- The set of all conic combinations of points in K



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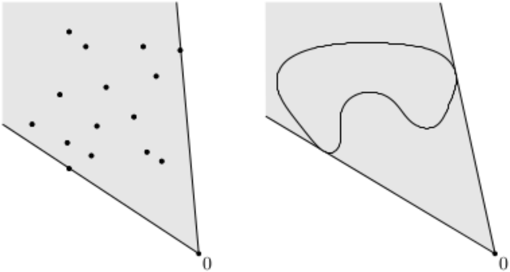
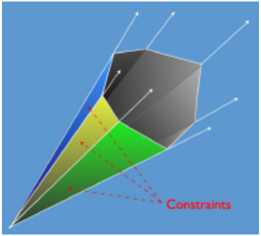


A set K is a convex cone if and only if conichull$(K) = K$.

Cones

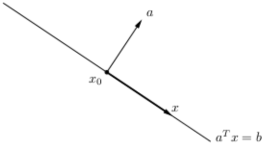
Polyhedral Cone

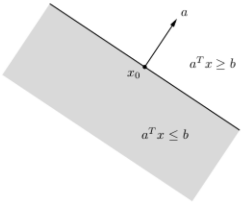
A cone is polyhedral if it is the set of solutions to a finite set of homogeneous linear inequalities $Ax \leq 0$.

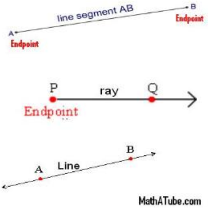


Outline

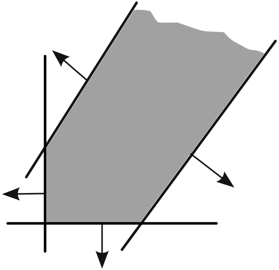
- Linear Subspace: Affine, Cone
- Hyperplane: Affine, cone if includes 0
- Halfspace: Cone if origin on boundary
- Line: Affine, cone if includes 0
- Ray: Cone if endpoint at 0
- Line segment



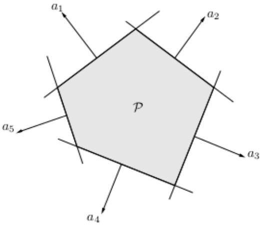




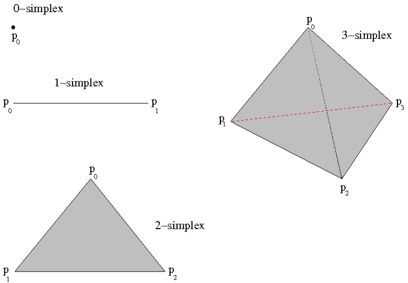
- Polyhedron: finite intersection of halfspaces



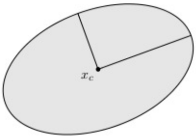
- Polytope: Bounded polyhedron



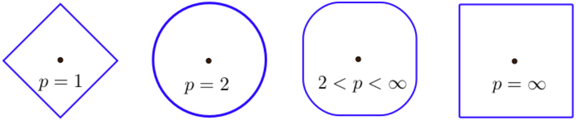
- Nonnegative Orthant $\mathbb{R}^n_+$: Polyhedral cone
- Simplex: convex hull of affinely independent points
  - Unit simplex: $x \succeq 0$, $\sum_i x_i \leq 1$
  - Probability simplex: $x \succeq 0$, $\sum_i x_i = 1$.



- Euclidean ball: $\{x : ||x - x_c||_2 \leq r\}$ for center x_c and radius r
- Ellipsoid: $\left\{x : (x - x_c)^T P^{-1}(x - x_c) \leq 1\right\}$ for symmetric $P \succeq 0$
  - Equivalently: $\{x_c + Au : ||u||_2 \leq 1\}$ for some linear map A



- Norm ball: $\{x : ||x - c|| \leq r\}$ for any norm $||.||$



The unit sphere for different metrics: $\|x\|_{l_p} = 1$ in $\mathbb{R}^2$.

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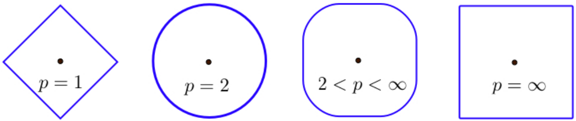
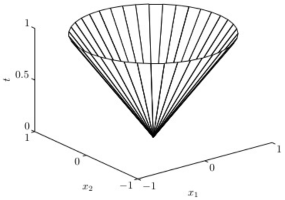
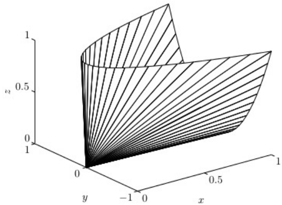


The unit sphere for different metrics: $\|x\|_{l_p} = 1$ in $\mathbb{R}^2$.

- Norm cone: $\{(x, r) : ||x|| \leq r\}$
- Cone of symmetric positive semi-definite matrices M
  - Symmetric matrix $A \succeq 0$ iff $x^T A x \geq 0$ for all x

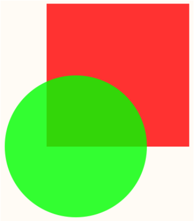




Outline

Intersection

The intersection of two convex sets is convex. This holds for the intersection of an infinite number of sets.



Examples

- Polyhedron: intersection of halfspaces
- PSD cone: intersection of linear inequalities $z^T A z \geq 0$, for all $z \in \mathbb{R}^n$.

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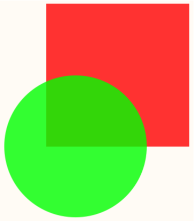


Examples

- Polyhedron: intersection of halfspaces
- PSD cone: intersection of linear inequalities $z^T A z \geq 0$, for all $z \in \mathbb{R}^n$.

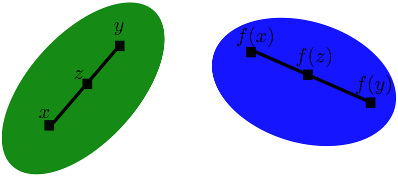
In fact, we will see that every closed convex set is the intersection of a (possibly infinite) set of halfspaces.

Affine Maps

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function (i.e. $f(x) = Ax + b$), then

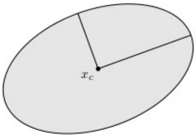
- $f(S)$ is convex whenever $S \subseteq \mathbb{R}^n$ is convex
- $f^{-1}(T)$ is convex whenever $T \subseteq \mathbb{R}^m$ is convex

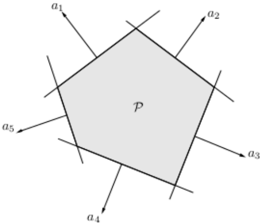
$$
\begin{aligned}
f(\theta x + (1-\theta)y) &= A(\theta x + (1-\theta)y) + b \\
&= \theta(Ax + b) + (1-\theta)(Ay + b)) \\
&= \theta f(x) + (1-\theta)f(y)
\end{aligned}
$$



Examples

- An ellipsoid is image of a unit ball after an affine map
- A polyhedron $Ax \preceq b$ is inverse image of nonnegative orthant under $f(x) = b - Ax$





Perspective Function

Let $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be $P(x, t) = x/t$.

- $P(S)$ is convex whenever $S \subseteq \mathbb{R}^{n+1}$ is convex
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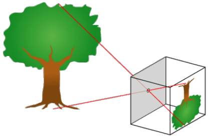

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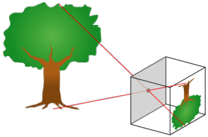


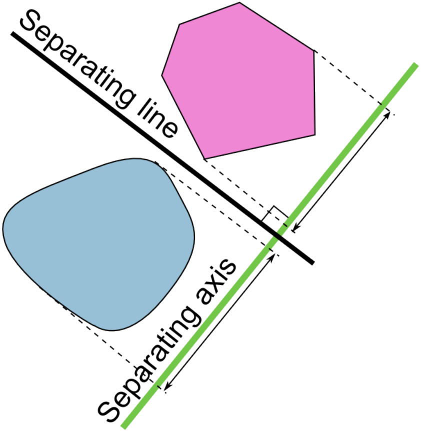
Generalizes to linear fractional functions $f(x) = \frac{Ax+b}{c^T x+d}$

- Composition of perspective with affine.

Outline

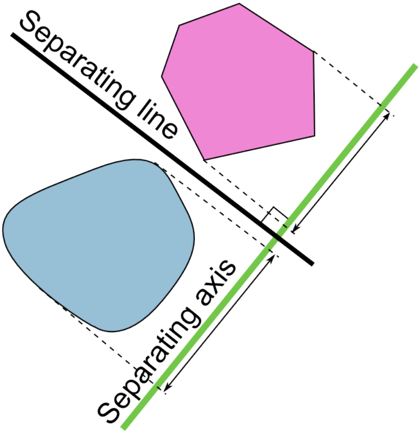
Separating Hyperplane Theorem

If $A, B \subseteq \mathbb{R}^n$ are disjoint convex sets, then there is a hyperplane weakly separating them. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^\intercal x \leq b$ for every $x \in A$ and $a^\intercal y \geq b$ for every $y \in B$.



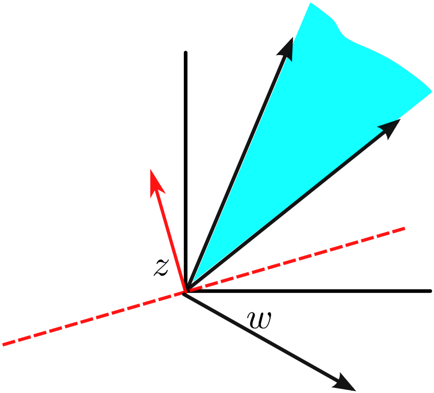
Separating Hyperplane Theorem (Strict Version)

If $A, B \subseteq \mathbb{R}^n$ are disjoint closed convex sets, and at least one of them is compact, then there is a hyperplane strictly separating them. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^\intercal x < b$ for every $x \in A$ and $a^\intercal y > b$ for every $y \in B$.



Farkas' Lemma

Let K be a closed convex cone and let $w \notin K$. There is $z \in \mathbb{R}^n$ such that $z^\intercal x \geq 0$ for all $x \in K$, and $z^\intercal w < 0$.



Supporting Hyperplane

Supporting Hyperplane Theorem.

If $S \subseteq \mathbb{R}^n$ is a closed convex set and y is on the boundary of S, then there is a hyperplane supporting S at y. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^\intercal x \leq b$ for every $x \in S$ and $a^\intercal y = b$.