CS675: Convex and Combinatorial Optimization
Fall 2023
The Ellipsoid Algorithm

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History and Basics

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- Inefficient in practice, has been surpassed by newer interior point methods.
- Theoretically most powerful, with deep consequences for complexity and optimization
  - Polynomial-time algorithm for linear programming
  - Polynomial-time algorithm for approximate convex optimization under mild conditions
  - Equivalence of separation and optimization

Outline

Convex Feasibility Problem

Given as input the following

- A description of a compact convex set $K \subseteq \mathbb{R}^n$
- An ellipsoid $E(c, Q)$ (typically a ball) containing K
- A rational number $R > 0$ satisfying $\mathbf{vol}(E) \leq R$.
- A rational number $r > 0$ such that if K is nonempty, then $\mathbf{vol}(K) \geq r$.

Find a point $x \in K$ or declare that K is empty.

- Note: convex optimization reduces to checking feasibility by binary search
  - We will explain later how the parameters r, R, E figure into this
- Description of K: any description that admits an efficient implementation of a separation oracle

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separting x from K.

- i.e. a vector $h \in \mathbb{R}^n$ with $\langle h, y \rangle < \langle h, x \rangle$ for all $y \in K$.
- Equivalently, K is contained in the open halfspace

$$H(h, x) = \{y : \langle h, y \rangle < \langle h, x \rangle\}$$

with x at its boundary.

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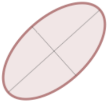
- The positive semi-definite cone S_n^+: Let $H = -vv^\intercal$, where v is an eigenvector corresponding to a negative eigenvalue.

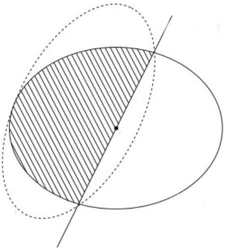
Recall:Ellipsoids



- Unit ball $B(0,1) = \{x \in \mathbb{R}^n : \langle x, x \rangle \leq 1\}$.
- Ellipsoid: the result of applying some affine transformation $x \to Lx + c$ to $B(0,1)$.
- We will focus on full dimensional case, with nonsingular L
- Gives the set $\left\{x : (x-c)^\intercal L^{-\intercal} L^{-1}(x-c) \leq 1\right\}$
- Conventional to take $Q = LL^\intercal$, which is positive definite, and write

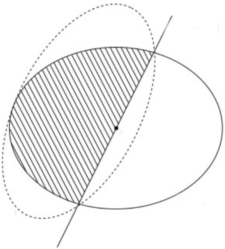
$$E(c,Q) = \left\{x : (x-c)^\intercal Q^{-1}(x-c) \leq 1\right\}$$

- Can calculate volume easily: $\mathbf{vol}(E(c,Q)) = \sqrt{\det Q}\, \mathbf{vol}(B(0,1))$
- Note: since $Q \succeq 0$ has a PSD square root, can take L to be PSD as well



Ellipsoid Method

1. Start with initial ellipsoid $E = E(c, Q) \supseteq K$
2. Using the separation oracle, check if the center $c \in K$.
   - If so, terminate and output c.
   - Otherwise, we get a separating hyperplane h such that K is contained in the half-ellipsoid $E \bigcap \{y : \langle h, y \rangle \leq \langle h, c \rangle\}$
3. Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
4. If $\mathbf{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return "empty".



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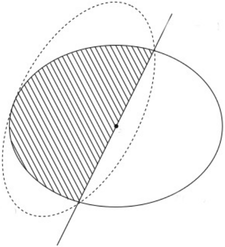

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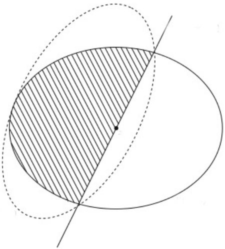

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Correctness

Theorem

If the ellipsoid algorithm terminates, then it either outputs $x \in K$ or correctly declares that K is empty.

- Algorithm outputs some x only if it was certified in K by the separation oracle
- $E \supseteq K$ is maintained throughout the algorithm, by definition
- We are promised $\mathbf{vol}(K) < r$ only if $K = \emptyset$
- If algorithm outputs empty then $r > \mathbf{vol}(E) \geq \mathbf{vol}(K)$, and $K = \emptyset$.

Runtime

To prove that the ellipsoid method runs in polynomial time, we need the following lemma

Lemma

- The minimum volume ellipsoid E' containing half of another ellipsoid E can be computed in $\mathrm{poly}(n)$ operations.
- Moreover, the volume of E' is smaller than the volume of E by a factor of at least $e^{\frac{1}{2(n+1)}} \approx 1 + \frac{1}{2(n+1)}$.

This is tricky to show, but there is an explicit, easy-to-compute, closed form for the matrix and center of E' in terms of the matrix and center of E. (See notes for proof)

Runtime

Theorem

The ellipsoid method terminates after $2(n+1)\ln\frac{R}{r}$ iterations. Moreover, each iteration can be implemented using $\mathrm{poly}(n)$ operations and a single call to the separation oracle.

- Volume of working ellipsoid starts off at R
- Decreases by a factor of $e^{\frac{1}{2(n+1)}}$ each iteration.
- After $2(n+1)\ln\frac{R}{r}$ interations, volume is less than r, triggering termination.

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- In each iteration, we
  - Call the separation oracle
  - Compute the minimum volume ellipsoid ($\mathrm{poly}(n)$ time by lemma)
  - Compute the volume of E' (reduces to determinant computation).