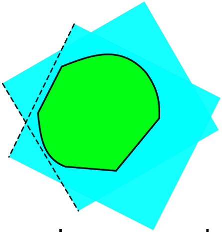
CS675: Convex and Combinatorial Optimization
Fall 2023
Geometric Duality of Convex Sets and Functions

Instructor: Shaddin Dughmi

Outline

1. Convexity and Duality
2. Duality of Convex Sets
3. Duality of Convex Functions

Duality Correspondances



There are two equivalent ways to represent a convex set

- The family of points in the set (standard or “primal” representation)
- The set of halfspaces containing the set (“dual” representation)

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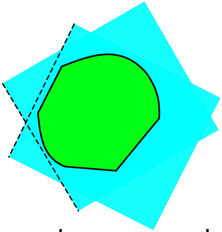

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This equivalence between the two representations gives rise to a variety of “duality” relationships among convex sets, cones, and functions.

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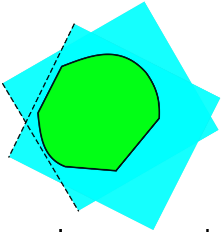

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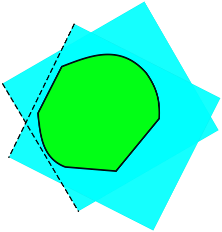
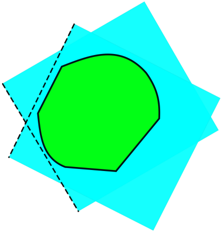
Definition

“Duality” is a woefully overloaded mathematical term for a relation that groups elements of a set into “dual” pairs.



Theorem

A closed convex set S is the intersection of all closed halfspaces $\mathcal{H}$ containing it.



Theorem

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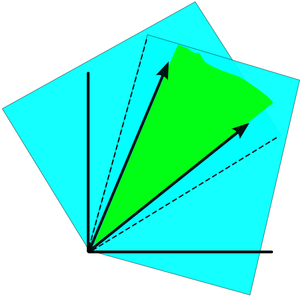
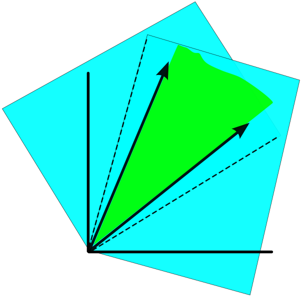
Proof

- Clearly, $S \subseteq \bigcap_{H \in \mathcal{H}} H$
- To prove equality, consider $x \notin S$
- By the separating hyperplane theorem, there is a hyperplane separating S from x
- Therefore there is $H \in \mathcal{H}$ with $x \notin H$, hence $x \notin \bigcap_{H \in \mathcal{H}} H$



Theorem

A closed convex cone K is the intersection of all closed homogeneous halfspaces $\mathcal{H}$ containing it.

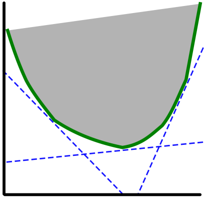


Theorem

A closed convex cone K is the intersection of all closed homogeneous halfspaces $\mathcal{H}$ containing it.

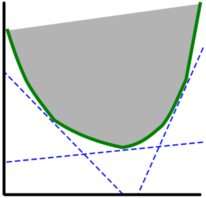
Proof

- For every non-homogeneous halfspace $\langle a, x \rangle \leq b$ containing K, the smaller homogeneous halfspace $\langle a, x \rangle \leq 0$ contains K as well.
- Therefore, can discard non-homogeneous halfspaces without changing the intersection



Theorem

A convex function is the point-wise supremum of all affine functions under-estimating it everywhere.



Theorem

A convex function is the point-wise supremum of all affine functions under-estimating it everywhere.

Proof

- $\mathbf{epi}\, f$ convex, therefore is the intersection of family of halfspaces $\mathcal{H}$
- Each $h \in \mathcal{H}$ can be written as $\langle a, x \rangle - t \leq b$, for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. (Why?)
  - Constrains $(x, t) \in \mathbf{epi}\, f$ to $\langle a, x \rangle - b \leq t$
- $f(x)$ is the lowest t s.t. $(x, t) \in \mathbf{epi}\, f$
- Therefore, $f(x)$ is the point-wise maximum of $\langle a, x \rangle - b$ over all halfspaces $h(a, b) \in \mathcal{H}$.

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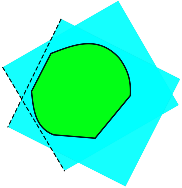
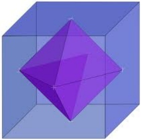
Polar Duality of Convex Sets





One way of representing all the halfspaces containing a convex set.

Polar

Let $S \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The polar of S is defined as follows:

$$S^\circ = \{y : \langle y, x \rangle \leq 1 \text{ for all } x \in S\}$$

Note

- Every halfspace $\langle a, x \rangle \leq b$ with $b \neq 0$ can be written as a "normalized" inequality $\langle y, x \rangle \leq 1$, by dividing by b.
- S° can be thought of as the normalized representations of halfspaces containing S.

$$S^\circ = \{y : \langle y, x\rangle \le 1 \text{ for all } x \in S\}$$

Properties of the Polar

1. $S^{\circ\circ} = S$
2. S° is a closed convex set containing the origin
3. When 0 is in the interior of S, then S° is bounded.

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Properties of the Polar

1. $S^{\circ\circ} = S$
2. S° is a closed convex set containing the origin
3. When 0 is in the interior of S, then S° is bounded.

2. Follows from representation as intersection of halfspaces
3. S contains an ϵ-ball centered at the origin, so S° is contained in the $\frac{1}{\epsilon}$ ball centered at the origin.
   - Take $y \in S^\circ$
   - $x := \epsilon \frac{y}{||y||_2} \in S$
   - $1 \ge \langle y, x\rangle = \epsilon||y||_2$, so $||y||_2 \le \frac{1}{\epsilon}$

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$$S^{\circ\circ} = \{x : \langle x, y\rangle \leq 1 \text{ for all } y \in S^\circ\}$$

1.
   - $S \subseteq S^{\circ\circ}$ is easy: $\widehat{x} \in S \implies \forall y \in S^\circ \ \langle \widehat{x}, y\rangle \leq 1 \implies \widehat{x} \in S^{\circ\circ}$
   - Take $\widehat{x} \notin S$, by SSHT and $0 \in S$, there is a halfspace $\langle z, x\rangle \leq 1$ containing S but not $\widehat{x}$ (i.e. $\langle z, \widehat{x}\rangle > 1$)
   - $z \in S^\circ$, therefore $\widehat{x} \notin S^{\circ\circ}$

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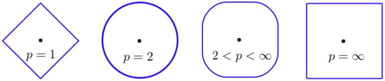
Properties of the Polar

1. $S^{\circ\circ} = S$
2. S° is a closed convex set containing the origin
3. When 0 is in the interior of S, then S° is bounded.

Note

When S is non-convex, $S^\circ = (convexhull(S))^\circ$, and $S^{\circ\circ} = convexhull(S)$.

Examples

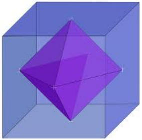


The unit sphere for different metrics: $\|x\|_{l_p} = 1$ in $\mathbb{R}^2$.

Norm Balls

- The polar of the Euclidean unit ball is itself (we say it is self-dual)
- The polar of the 1-norm ball is the ∞-norm ball
- More generally, the p-norm ball is dual to the q-norm ball, where $\frac{1}{p} + \frac{1}{q} = 1$

Examples

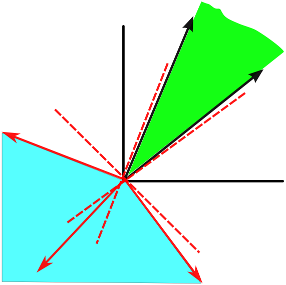


Polytopes

Given a polytope P represented as $Ax \preceq \vec{1}$, the polar P° is the convex hull of the rows of A.

- Facets of P correspond to vertices of P°.
- Dually, vertices of P correspond to facets of P°.

Polar Duality of Convex Cones



Polar duality takes a simplified form when applied to cones

Polar

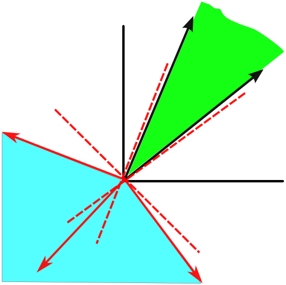
The polar of a closed convex cone K is given by

$$K^\circ = \{y : \langle y, x\rangle \leq 0 \text{ for all } x \in K\}$$

Note

- $\forall x \in K \ \langle y, x\rangle \leq 1 \iff \forall x \in K \ \langle y, x\rangle \leq 0$
- K° represents all homogeneous halfspaces containing K.

Polar Duality of Convex Cones



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Dual Cone

By convention, $K^* = -K^\circ$ is referred to as the dual cone of K.

$$K^* = \{y : \langle y, x \rangle \geq 0 \text{ for all } x \in K\}$$

$$K^\circ = \{y : \langle y, x\rangle \leq 0 \text{ for all } x \in K\}$$

Properties of the Polar Cone

1. $K^{\circ\circ} = K$
2. K° is a closed convex cone

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Properties of the Polar Cone

1. $K^{\circ\circ} = K$
2. K° is a closed convex cone

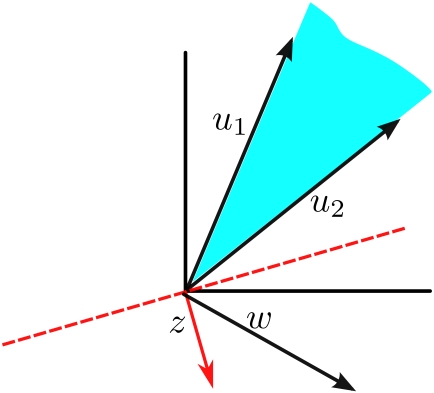
1. Same as before
2. Intersection of homogeneous halfspaces

Examples

- The polar of a subspace is its orthogonal complement
- The polar cone of the nonnegative orthant is the nonpositive orthant
  - Self-dual
- The polar of a polyhedral cone $Ax \preceq 0$ is the conic hull of the rows of A
- The polar of the cone of positive semi-definite matrices is the cone of negative semi-definite matrices
  - Self-dual

Recall: Farkas' Lemma

Let K be a closed convex cone and let $w \notin K$. There is $z \in \mathbb{R}^n$ such that $\langle z, x \rangle \leq 0$ for all $x \in K$, and $\langle z, w \rangle > 0$.



Equivalently: there is $z \in K^\circ$ with $\langle z, w \rangle > 0$.

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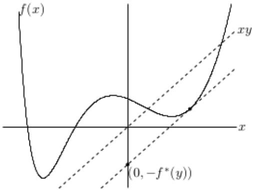
Conjugation of Convex Functions



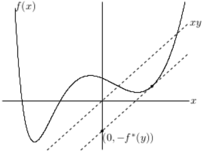
Conjugate

For a function $f : \mathbb{R}^n \to \mathbb{R} \bigcup \{\infty\}$, the conjugate of f is

$$f^*(y) = \sup_x (\langle y, x \rangle - f(x))$$

Note

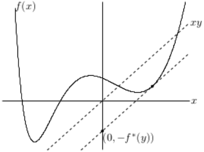
- $f^*(y)$ is the minimal value of β such that the affine function $\langle y, x \rangle - \beta$ underestimates $f(x)$ everywhere.
- Equivalently, the distance we need to lower the hyperplane $\langle y, x \rangle - t = 0$ in order to get a supporting hyperplane to $\mathbf{epi}\, f$.
- $\langle y, x \rangle - t = f^*(y)$ are the supporting hyperplanes of $\mathbf{epi}\, f$



$$f^*(y) = \sup_x(\langle y, x\rangle - f(x))$$

Properties of the Conjugate

1. $f^{**} = f$ when f is convex
2. f^* is a convex function
3. $xy \leq f(x) + f^*(y)$ for all $x, y \in \mathbb{R}^n$ (Fenchel's Inequality)



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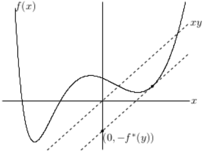
2. Supremum of affine functions of y
3. By definition of $f^*(y)$



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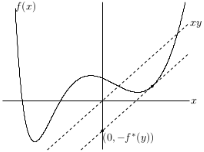
1. - $f^{**}(x) = \sup_y \langle y, x\rangle - f^*(y)$ by definition



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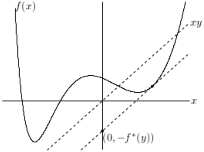
   - $f^{**}(x) = \sup_y \langle y, x\rangle - f^*(y)$ by definition
   - For fixed y, $f^*(y)$ is minimal β such that $\langle y, x\rangle - \beta$ underestimates f.



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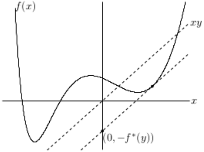
   - For fixed y, $f^*(y)$ is minimal β such that $\langle y, x\rangle - \beta$ underestimates f.
   - Therefore $f^{**}(x)$ is the maximum, over all y, of affine underestimates $\langle y, x\rangle - \beta$ of f



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   - Therefore $f^{**}(x)$ is the maximum, over all y, of affine underestimates $\langle y, x \rangle - \beta$ of f
   - By our earlier characterization, this is equal to f when f is convex.

Examples

- Affine function: $f(x) = ax + b$. Conjugate has $f^*(a) = -b$, and ∞ elsewhere
- $f(x) = x^2/2$ is self-conjugate
- Exponential: $f(x) = e^x$. Conjugate has $f^*(y) = y \log y - y$ for $y \in \mathbb{R}_+$, and ∞ elsewhere.
- Convex Quadratic: $f(x) = \frac{1}{2}x^\intercal Q x$ with Q positive definite. Conjugate is $f^*(y) = \frac{1}{2}y^\intercal Q^{-1} y$
- Log-sum-exp: $f(x) = \log(\sum_i e^{x_i})$. Conjugate has $f^*(y) = \sum_i y_i \log y_i$ for $y \succeq 0$ and $\langle \vec{1}, y \rangle = 1$, ∞ otherwise.