CS675: Convex and Combinatorial Optimization
Fall 2023
Introduction

Instructor: Shaddin Dughmi

Outline

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1. Course Overview

2. Administrivia

Mathematical Optimization

The task of selecting the "best" configuration of a set of variables from a "feasible" set of configurations.

$$\begin{array}{ll} \text{minimize (or maximize)} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

- Terminology: decision variable(s), objective function, feasible set, optimal solution, optimal value
- Two main classes: continuous and combinatorial

Continuous Optimization Problems

Optimization problems where feasible set $\mathcal{X}$ is a connected subset of Euclidean space, and f is a continuous function.

- Instances typically formulated as follows.

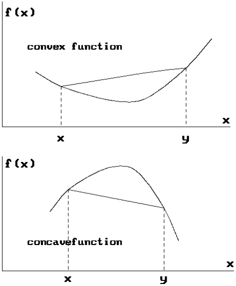
$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq b_i, \quad \text{for } i \in \mathcal{C}. \end{array}$$

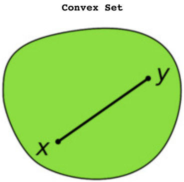
- Objective function $f : \mathbb{R}^n \to \mathbb{R}$.
- Constraint functions $g_i : \mathbb{R}^n \to \mathbb{R}$. The inequality $g_i(x) \leq b_i$ is the i'th constraint.
- In general, intractable to solve efficiently (NP hard)

Convex Optimization Problem

A continuous optimization problem where f is a convex function on $\mathcal{X}$, and $\mathcal{X}$ is a convex set.

- Convex function: $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in \mathcal{X}$ and $\alpha \in [0,1]$
- Convex set: $\alpha x + (1-\alpha)y \in \mathcal{X}$, for all $x, y \in \mathcal{X}$ and $\alpha \in [0,1]$
- Convexity of $\mathcal{X}$ implied by convexity of g_i's
- For maximization problems, f should be concave
- Typically solvable efficiently (i.e. in polynomial time)
- Encodes optimization problems from a variety of application areas



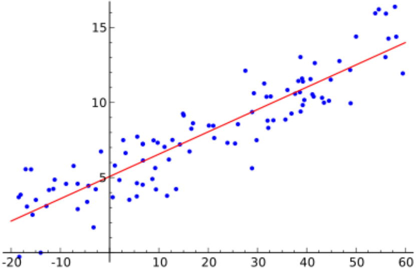


Convex Optimization Example: Least Squares Regression

Given a set of measurements $(a_1, b_1), \ldots, (a_m, b_m)$, where $a_i \in \mathbb{R}^n$ is the i'th input and $b_i \in \mathbb{R}$ is the i'th output, find the linear function $f : \mathbb{R}^n \to \mathbb{R}$ best explaining the relationship between inputs and outputs.

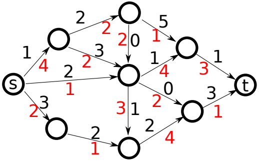
- $f(a) = \langle x, a \rangle$ for some $x \in \mathbb{R}^n$
- Least squares: minimize mean-square error.

$$\text{minimize} \quad ||Ax - b||_2^2$$



Convex Optimization Example: Minimum Cost Flow

Given a directed network $G = (V, E)$ with cost $c_e \in \mathbb{R}_+$ per unit of traffic on edge e, and capacity d_e, find the minimum cost routing of r divisible units of traffic from s to t.



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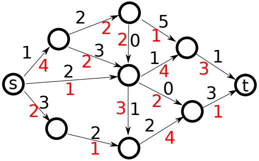

$$
\begin{array}{lll}
\text{minimize} & \sum_{e \in E} c_e x_e & \\
\text{subject to} & \sum_{e \leftarrow v} x_e = \sum_{e \rightarrow v} x_e, & \text{for } v \in V \setminus \{s, t\}. \\
& \sum_{e \leftarrow s} x_e = r & \\
& x_e \leq d_e, & \text{for } e \in E. \\
& x_e \geq 0, & \text{for } e \in E.
\end{array}
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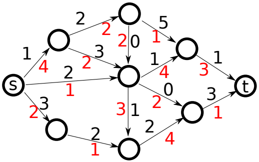

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\end{array}
$$

Generalizes to traffic-dependent costs. For example $c_e(x_e) = a_e x_e^2 + b_e x_e + c_e$.

Combinatorial Optimization

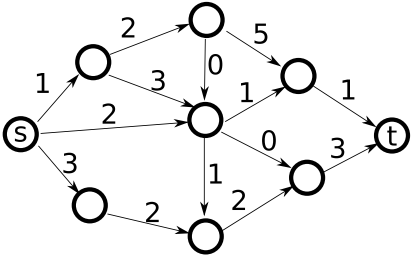
Combinatorial Optimization Problem

An optimization problem where the feasible set $\mathcal{X}$ is finite.

- e.g. $\mathcal{X}$ is the set of paths in a network, assignments of tasks to workers, etc...
- Again, NP-hard in general, but many are efficiently solvable (either exactly or approximately)

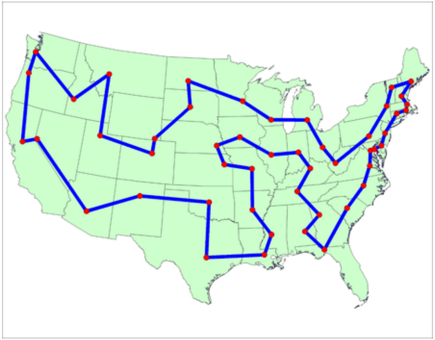
Combinatorial Optimization Example: Shortest Path

Given a directed network $G = (V, E)$ with cost $c_e \in \mathbb{R}_+$ on edge e, find the minimum cost path from s to t.



Combinatorial Optimization Example: Traveling Salesman Problem

Given a set of cities V, with $d(u, v)$ denoting the distance between cities u and v, find the minimum length tour that visits all cities.

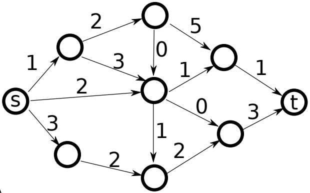


Continuous vs Combinatorial Optimization

- Some optimization problems are best formulated as one or the other
- Many problems, particularly in computer science and operations research, can be formulated as both
- This dual perspective can lead to structural insights and better algorithms

Example: Shortest Path

The shortest path problem can be encoded as a minimum cost flow problem, using distances as the edge costs, unit capacities, and desired flow rate 1



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\text{minimize} & \sum_{e \in E} c_e x_e & \\
\text{subject to} & \sum_{e \leftarrow v} x_e = \sum_{e \rightarrow v} x_e, & \text{for } v \in V \setminus \{s, t\}. \\
& \sum_{e \leftarrow s} x_e = {\color{red}1} & \\
& x_e \leq {\color{red}1}, & \text{for } e \in E. \\
& x_e \geq 0, & \text{for } e \in E.
\end{array}
$$

The optimum solution of the (linear) convex program above will assign flow only on a shortest path.

Course Goals

- Recognize and model convex optimization problems, and develop a general understanding of the relevant algorithms.
- Formulate combinatorial optimization problems as convex programs
- Use both the discrete and continuous perspectives to design algorithms and gain structural insights for optimization problems

Who Should Take this Class

- Anyone planning to do research in the design and analysis of algorithms
  - Convex and combinatorial optimization have become an indispensible part of every algorithmist's toolkit
- Students interested in theoretical machine learning and AI
  - Convex optimization underlies much of machine learning
  - Submodularity has recently emerged as an important abstraction for feature selection, active learning, planning, and other applications
- Anyone else who solves or reasons about optimization problems: electrical engineers, control theorists, operations researchers, economists ...
  - If there are applications in your field you would like to hear more about, let me know.

Who Should Not Take this Class

- You don’t satisfy the prerequisites “in practice”
- You are looking for a “cookbook” of optimization algorithms, and/or want to learn how to use Gurobi, CPLEX, CVX, etc
    - This is a THEORY class
    - We will bias our attention towards simple yet theoretically insightful algorithms and questions
    - We will not write code

Course Outline

- Weeks 1-5: Convex optimization basics and duality theory
- Weeks 6-7: Combinatorial problems posed as linear and convex programs
- Weeks 8-9: Algorithms for convex optimization
- Weeks 10-11: Matroid theory and optimization
- Weeks 12-13: Submodular Function optimization
- Week 14: Semidefinite programming and constraint satisfaction problems
- Week 15: Additional topics

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Basic Information

- Lecture time: Mondays and Wednesdays 4:00pm - 5:50pm
- Lecture place: DMC 260
- Instructor: Shaddin Dughmi
  - Email: shaddin@usc.edu
  - Office: SAL 234
  - Office Hours: TBD
- TA: Neel Patel
  - Email: neelbpat@usc.edu
  - Office Hours: TBD
- Course Homepage: https://viterbi-web.usc.edu/ shaddin/cs675fa23/index.html
- References: Convex Optimization by Boyd and Vandenberghe, and Combinatorial Optimization by Korte and Vygen. (Available online through USC libraries.)
- Additional References: Schrijver, Luenberger and Ye (available online through USC libraries)

Prerequisites

- Mathematical maturity: Be good at proofs, at the graduate level.
- Linear algebra at advanced undergrad / beginning grad level
- Exposure to algorithms or optimization at advanced undergrad / beginning grad level
  - CS570 or equivalent, or
  - CS270 and you did really well

Requirements and Grading

- This is an advanced elective class, so grade is not the point.
  - I assume you want to learn this stuff.
- 6-8 homeworks, 75% of grade.
  - Proof based.
  - Challenging.
  - Discussion allowed, even encouraged, but must write up solutions independently.
- Research project worth 25% of grade: Can be theoretical, applied, or anything in between.
- 6 late days allowed total (use in integer amounts)