# CS675: Convex and Combinatorial Optimization
# Fall 2023
# The Simplex Algorithm

Instructor: Shaddin Dughmi

# History and Basics of the Simplex Algorithm

- First methodical procedure for solving linear programs
- Developed by George Dantzig in 1947
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- Efficient in practice, leading to conjectures that it runs in polynomial time
- In 1972, Klee and Minty exhibited worst-case examples that take exponential time, at least for some of the most popular simplex pivot rules
- This spurred development of the Ellipsoid method, interior point methods, ...

# Outline

# Linear Programming

We consider a standard form LP written as follows for convenience

$$\begin{array}{ll} \text{maximize} & c^{\mathsf{T}}x \\ \text{subject to} & Ax \preceq b \end{array}$$

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- Degeneracy: a vertex with $> n$ tight inequalities
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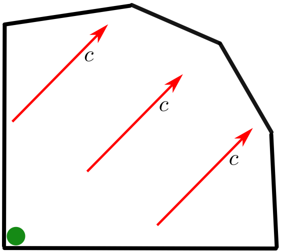
- Incidentally, algorithm will produce optimal dual $y^*$ as well.

# Recall: Physical Interpretation of LP

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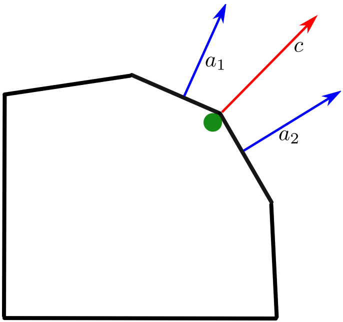


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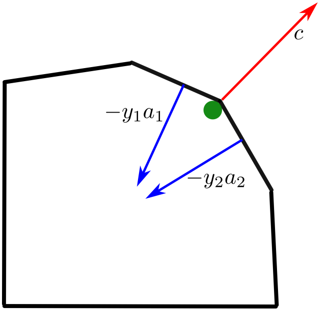


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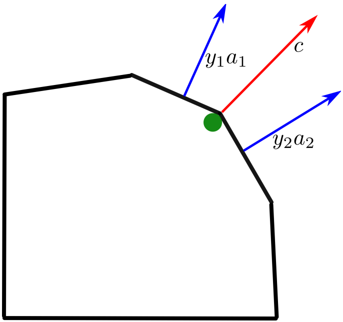


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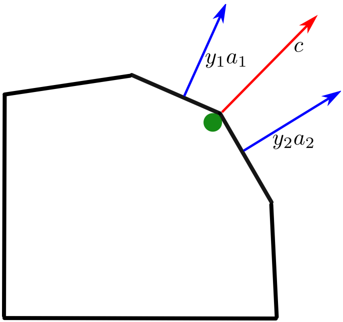


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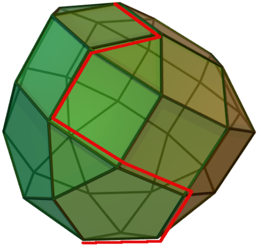
- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball for some $y_i \geq 0$
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.
- At optimality, only the walls adjacent to the ball push (Complementary Slackness)
  - Necessary and sufficient for optimality, given dual-feasible $y$

# Informal Description



- Starts at initial vertex $x = x_0$
- While $x$ is not optimal, move to a neighbouring vertex $x'$ with $cx' > cx$.

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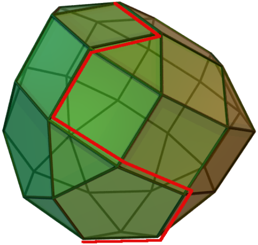

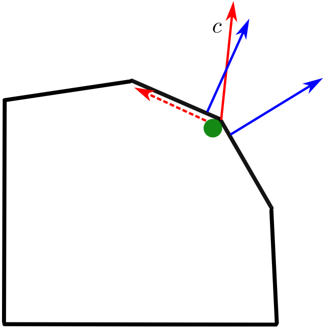


- Starts at initial vertex $x = x_0$
- While $x$ is not optimal, move to a neighbouring vertex $x'$ with $cx' > cx$.
  - Either $c$ is in the cone defined by tight constraints at $x$, in which case $x$ is optimal by complementary slackness
  - Or else can improve $cx$ by moving along an edge (1-d face)

## Simplex Method

- **Input:** vertex $x = x_0$
- **Output:** Optimal vertex $x^*$ and complementary dual $y^*$, or unbounded

**Repeat the following:**

1. Write $c^\intercal = y^\intercal A$, where $y_i \neq 0$ only for $n$ tight constraints $a_i x = b_i$.
2. If $y \succeq 0$ then **stop and return** $(x, y)$, else
3. Choose $i$ with $y_i < 0$, and let $\vec{d}$ be s.t. $A_{T \setminus \{i\}} d = 0$ and $a_i d = -1$.
4. If $x + \lambda d$ feasible for all $\lambda \geq 0$, **stop and return unbounded**, else
5. $x \leftarrow x + \lambda d$, for largest $\lambda \geq 0$ maintaining feasibility

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- Let $T$ be set of $n$ linearly independent rows which are tight at $x$.
- $y_T^\intercal A_T = c^\intercal$
- Gaussian elimination

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- $y$ is a dual satisfying complementary slackness with $x$
- Therefore both are optimal

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- Chosen so that moving in direction $d$ preserves tightness of constraints $T \setminus \{i\}$, and loosens constraint $i$.
- $A_T$ is full-rank, therefore $null(A_{T \setminus \{i\}})$ is a 1-dimensional subspace which is not normal to $a_i$
- Choose $d \in null(A_{T \setminus \{i\}})$ appropriately.
- Moving in direction $d$ improves objective: $c^\intercal d = y^\intercal A d = y_i a_i d > 0$

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- i.e. $Ad \preceq 0$

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- $\lambda = \min \left\{ \frac{b_j - a_j x}{a_j d} : j \in [m], a_j d > 0 \right\}$
- $j$ achieving this minimum is a new tight constraint, replacing $i$.
- By nondegeneracy assumption, $\lambda > 0$

# Outline

# Correctness

## Claim

If the simplex algorithm terminates, then it correctly outputs either an optimal primal/dual pair or unbounded.

- Primal feasibility of $x$ is maintained throughout
- Returns $(x, y)$ only if $y$ is dual feasible and satisfies complementary slackness
  - $x$ and $y$ are both optimal
- Returns unbounded only if there is a direction $d$ with $c^\intercal d > 0$ and $Ad \preceq 0$.

# Termination in the Absence of Degeneracy

## Claim

In the absence of degenerate vertices, the simplex algorithm terminates in a finite number of steps, at most $\binom{m}{n} \leq 2^m$.

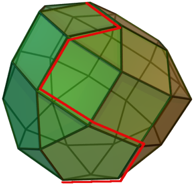
- There are at most $\binom{m}{n}$ distinct vertices in the polyhedron
- In the absence of degeneracy, the simplex algorithm does not repeat a vertex
  - In each iteration, moves along an edge in direction $d$, in total $\lambda d$
  - We saw: $c^\intercal d > 0$, and $\lambda > 0$.
  - Objective <u>strictly</u> improves each iteration

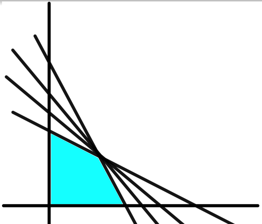
# Pivot Rules

## Note

The algorithm we presented was not fully specified

- When multiple neighboring vertices are improving, which one should we choose so as to terminate as quickly as possible?
- In the presence of degeneracy, how should we identify the next (geometric) vertex so as to guarantee termination?
  - We maintain $n$ tight and linearly independent constraints $T$, to be thought of as an algebraic representation of a vertex (aka a basic feasible solution (BFS))
  - When many algebraic representations are possible of a single geometric vertex, unclear how to identify the next geometric vertex.





# Pivot Rules

Both concerns are addressed by the use of a pivot rule, which determines the order in which we examine algebraic vertices.

## Pivot rule

A rule for selecting which $i$ leaves $T$, and which $j$ enters $T$, when multiple choices are possible either because of multiple improving neighbors or degeneracy. Examples:

- Bland's rule: Choose lowest indexed $i$, then lowest indexed $j$
- Lexicographic: Maintain an order over rows, and move from $T$ to the lexicographically smallest possible $T'$.
- Perturbation: perturb entries of $b$ by a small value to remove degeneracy. This perturbation can be purely symbolic.

# Runtime and Termination

- Many pivot rules, like the ones we mentioned, have been shown to never cycle over algebraic vertices
  - Guarantees termination in general, even in the presence of degeneracies
  - See book and notes for proofs.
- However, no pivot rules have been shown to guarantee a polynomial number of pivots
  - Even if no degeneracies.
- In 1972, Klee and Minty exhibited a family of examples that lead to exponential worst-case runtime for some widely-used pivot rules

# Runtime and Termination

Nevertheless, one explanation as to the efficiency of the simplex algorithm in practice is through smoothed complexity

**Theorem (Spielman & Teng '01)**

*The simplex algorithm has polynomial smoothed complexity.*

- Model of input:
  - $A$, $b$, $c$ chosen arbitrarily (worst case)
  - Then subjected to small gaussian noise with stddev $\sigma$ (relative to largest entry of $A, b, c$)
  - Interpretation: measurement error
- More optimistic than worst case, but not quite as optimistic as average case.
- Expected runtime is polynomial in $n$, $m$ and $\frac{1}{\sigma}$

# Runtime and Termination

## Open Question

Is there a pivot rule which guarantees a polynomial number of pivots of the simplex algorithm in the worst case?

Why is this important?

- Would yield a strongly polynomial algorithm for LP
- If true, resolves in the affirmative a classic open question in polyhedral combinatorics
  - Polynomial Hirsch Conjecture: Is the diameter of the edge-vertex graph of an $m$-facet polytope in $n$-dimensional space bounded by a polynomial in $n$ and $m$?

# Outline

# Initialization

**Solving a Linear Program via the Simplex Method**

- Phase I: Find a vertex $x_0$.
- Phase II: Run the simplex algorithm starting from $x_0$

- So far, we have looked only at phase II
- For phase I, we pose a different LP whose optimal solution is a vertex, if one exists

# Phase I

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- An optimal vertex of new LP (with $z = 0$) will correspond to some vertex $x_0$ of original LP

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- We need a starting vertex for new LP, this is easier!
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- Running simplex on new LP with starting vertex $(x_0', z_0)$, we get starting vertex $x_0$ for original LP.