CS675: Convex and Combinatorial Optimization
Fall 2023
Submodular Function Optimization

Instructor: Shaddin Dughmi

Outline

Introduction

- We saw how matroids form a class of feasible sets over which optimization of modular objectives is tractable
- If matroids are discrete analogues of convex sets, then submodular functions are discrete analogues of convex/concave functions
  - Submodular functions behave like convex functions sometimes (minimization) and concave other times (maximization)
- Today we will introduce submodular functions, go through some examples, and mention some of their properties

Set Functions

- A set function takes as input a set, and outputs a real number
  - Inputs are subsets of some ground set X
  - $f : 2^X \to \mathbb{R}$
- We will focus on set functions where X is finite, and denote $n = |X|$

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- We will focus on set functions where X is finite, and denote $n = |X|$
- Equivalently: map points in the hypercube $\{0,1\}^n$ to the real numbers
  - Can be plotted as 2^n points in $n+1$ dimensional space

Set Functions

- We have already seen modular set functions
  - There is a weight w_i for each $i \in X$, and a constant c, such that $f(S) = c + \sum_{i \in S} w_i$ for all sets $S \subseteq X$.
  - Discrete analogue of affine functions

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  - Discrete analogue of affine functions
  - Direct definition of modularity: $f(A) + f(B) = f(A \cap B) + f(A \cup B)$
- Submodular/supermodular functions are weak analogues to convex/concave functions (in no particular order!)
- Other possibly useful properties a set function may have:
  - Monotone increasing or decreasing
  - Nonnegative: $f(A) \geq 0$ for all $S \subseteq X$
  - Normalized: $f(\emptyset) = 0$.

Submodular Functions

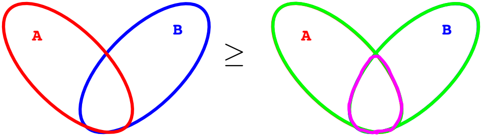
Definition 1

A set function $f : 2^X \to \mathbb{R}$ is submodular if and only if

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

for all $A, B \subseteq X$.

- "Uncrossing" two sets reduces their total function value



Submodular Functions

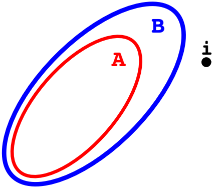
Definition 2

A set function $f : 2^X \to \mathbb{R}$ is submodular if and only if

$$f(B \cup \{i\}) - f(B) \leq f(A \cup \{i\}) - f(A))$$

for all $A \subseteq B \subseteq X$ and $i \notin B$.

- The marginal value of an additional element exhibits "diminishing marginal returns"
- Should remind of concavity: second "derivative" is negative



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Examples

Many common examples are monotone, normalized, and submodular.

Coverage Functions

- In general: X is a family of sets, and $f(S)$ is the "size" (cardinality or measure) of $\bigcup_{A \in S} A$
- Discrete special case: X the left hand side of a bipartite graph, and $f(S)$ is the total number of neighbors of S.

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The following two are examples of coverage functions

Probability

X is a set of probability events, and $f(S)$ is the probability at least one of them occurs.

Sensor Coverage

X is a family of locations in space you can place sensors, and $f(S)$ is the total area covered if you place sensors at locations $S \subseteq X$.

Examples

Social Influence

- X is the family of nodes in a social network
- A meme, idea, or product is adopted at a set of nodes S
- The idea propagates through the network through some random diffusion process
    - Many different models
- $f(S)$ is the expected number of nodes in the network which end up adopting the idea.

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Utility Functions

When X is a set of goods, $f(S)$ can represent the utility of an agent for a bundle of these goods. Utilities which exhibit diminishing marginal returns are natural in many settings.

Examples

Entropy

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Clustering Quality

X is the set of nodes in a graph G, and $f(S) = E(S)$ is the internal connectedness of cluster S.

- Supermodular

Examples

There are fewer examples of non-monotone submodular/supermodular functions, which are nontheless fundamental.

Graph Cuts

X is the set of nodes in a graph G, and $f(S)$ is the number of edges crossing the cut $(S, X \setminus S)$.

- Submodular
- Non-monotone.

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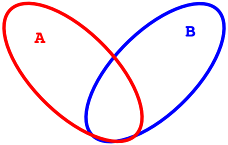
- Neither submodular nor supermodular
- However, maximizing it reduces to maximizing supermodular function $E(S) - \alpha|S|$ for various $\alpha > 0$ (binary search)

Equivalence of Both Definitions

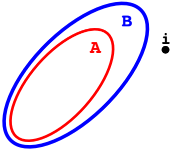
Definition 1

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$



Definition 2

$$f(B \cup \{i\}) - f(B) \leq f(A \cup \{i\}) - f(A))$$



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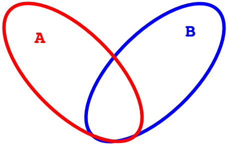

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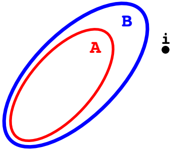

Definition 1 $\Rightarrow$ Definition 2

- To prove (2), let $A' = A\bigcup\{i\}$ and $B' = B$ and apply (1)

$$\begin{aligned} f(A\cup\{i\})+f(B) &= f(A')+f(B') \\ &\geq f(A'\cap B')+f(A'\cup B') \\ &= f(A)+f(B\cup\{i\}) \end{aligned}$$

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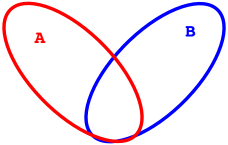

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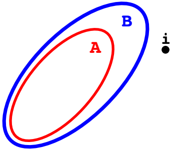

Definition 2 ⇒ Definition 1

- To prove (1), start with $A = B = A \bigcap B$ and repeatedly add elements to one but not the other
- At each step, (2) implies that the LHS of inequality (1) increases more than the RHS

Operations Preserving Submodularity

- Nonnegative-weighted combinations (a.k.a. conic combinations): If $f_1, \ldots, f_k$ are submodular, and $w_1, \ldots, w_k \geq 0$, then $g(S) = \sum_i w_i f_i(S)$ is also submodular
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Note

The minimum or maximum of two submodular functions is not necessarily submodular

Optimizing Submodular Functions

- As our examples suggest, optimization problems involving submodular functions are very common
- These can be classified on two axes: constrained/unconstrained and maximization/minimization

	Maximization	Minimization
Unconstrained	NP-hard $\frac{1}{2}$ approximation	Polynomial time via convex opt
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1. Introduction to Submodular Functions

2. Unconstrained Submodular Minimization
   - Definition and Examples
   - The Convex Closure and the Lovasz Extension
   - Wrapping up

3. Monotone Submodular Maximization s.t. a Matroid Constraint
   - Definition and Examples
   - Warmup: Cardinality Constraint
   - General Matroid Constraints

Recall: Optimizing Submodular Functions

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Problem Definition

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$$\begin{array}{ll} \text{minimize} & f(S) \\ \text{subject to} & S \subseteq X \end{array}$$

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Note: weakly polynomial. There are strongly polytime algorithms.

Examples

Minimum Cut

Given a graph $G = (V, E)$, find a set $S \subseteq V$ minimizing the number of edges crossing the cut $(S, V \setminus S)$.

- G may be directed or undirected.
- Extends to hypergraphs.

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Densest Subgraph

Given an undirected graph $G = (V, E)$, find a set $S \subseteq V$ maximizing the average internal degree.

- Reduces to supermodular maximization via binary search for the right density.

Continuous Extensions of a Set Function

Recall

A set function f on $X = \{1, \ldots, n\}$ can be thought of as a map from the vertices $\{0, 1\}^n$ of the n-dimensional hypercube to the real numbers.

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We will consider extensions of a set function to the entire hypercube.

Extension of a Set Function

Given a set function $f : \{0,1\}^n \to \mathbb{R}$, an extension of f to the hypercube $[0,1]^n$ is a function $g : [0,1]^n \to \mathbb{R}$ satisfying $g(x) = f(x)$ for every $x \in \{0,1\}^n$.

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Long story short…

We will exhibit an extension which is convex when f is submodular, and can be minimized efficiently. We will then show that minimizing it yields a solution to the submodular minimization problem.

The Convex Closure

Convex Closure

Given a set function $f : \{0,1\}^n \to \mathbb{R}$, the convex closure $f^- : [0,1]^n \to \mathbb{R}$ of f is the point-wise greatest convex function under-estimating f on $\{0,1\}^n$.

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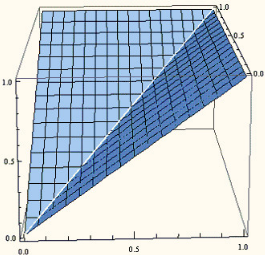
Geometric Intuition

What you would get by placing a blanket under the plot of f and pulling up.

$$f(\emptyset) = 0$$
$$f(\{1\}) = f(\{2\}) = 1$$
$$f(\{1,2\}) = 1$$

$$f^-(x_1, x_2) = \max(x_1, x_2)$$



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Claim

The convex closure exists for any set function.

Proof

- If $g_1, g_2 : [0,1]^n \to \mathbb{R}$ are convex under-estimators of f, then so is $\max\{g_1, g_2\}$
- Holds for infinite set of convex under-estimators
- Therefore $f^- = \max\{g : g \text{ is a convex underestimator of } f\}$ is the point-wise greatest convex underestimator of f.

Claim

The value of the convex closure f^- at $x \in [0,1]^n$ is the solution of the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \sum_{y \in \{0,1\}^n} \lambda_y f(y) \\ \text{subject to} & \sum_{y \in \{0,1\}^n} \lambda_y y = x \\ & \sum_{y \in \{0,1\}^n} \lambda_y = 1 \\ & \lambda_y \geq 0, \qquad \qquad \text{for } y \in \{0,1\}^n. \end{array}$$

Interpretation

- The minimum expected value of f over all distributions on $\{0,1\}^n$ with expectation x.
- Equivalently: the minimum expected value of f for a random set $S \subseteq X$ including each $i \in X$ with probability x_i.
- The upper bound on $f^-(x)$ implied by applying Jensen's inequality to every convex combination of $\{0,1\}^n$.

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Implications

- f^- is an extension of f.
- $f^-(x)$ has no “integrality gap”
  - For every $x \in [0,1]^n$, there is a random integer vector $y \in \{0,1\}^n$ such that $\mathbf{E}_y\, f(y) = f^-(x)$.
  - Therefore, there is an integer vector y such that $f(y) \leq f^-(x)$.

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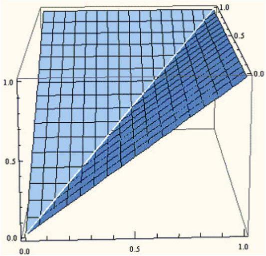
$f(\emptyset) = 0$
$f(\{1\}) = f(\{2\}) = 1$
$f(\{1,2\}) = 1$

When $x_1 \leq x_2$

$$\begin{aligned} f^-(x_1, x_2) &= x_1 f(\{1,2\}) \\ &+ (x_2 - x_1) f(\{2\}) \\ &+ (1 - x_2) f(\emptyset) \end{aligned}$$



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- Under-estimate: $OPT(x) = f(x)$ for $x \in \{0,1\}^n$
- Convex: The value of a minimization LP is convex in its right hand side constants (check)

Using the Convex Closure

Fact

The minimum of f^- is equal to the minimum of f, and moreover is attained at minimizers $y \in \{0,1\}^n$ of f.

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- Therefore $\min_{x \in [0,1]^n} f^-(x) \leq \min_{y \in \{0,1\}^n} f(y)$
- For every x, $f^-(x)$ is the expected value of $f(y)$, for a random variable $y \in \{0,1\}^n$ with expectation x.
- Therefore, $\min_{x \in [0,1]^n} f^-(x) \geq \min_{y \in \{0,1\}^n} f(y)$

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We reduced minimizing set function f to minimizing a convex function f^- over a convex set $[0,1]^n$. Are we done?

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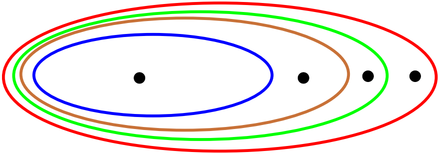
In general, it is hard to evaluate f^- efficiently, let alone its derivative. This is indispensible for convex optimization algorithms.

We will show that, when f is submodular, f^- is in fact equivalent to another extension which is easier to evaluate.

Chain Distributions

Chain Distribution

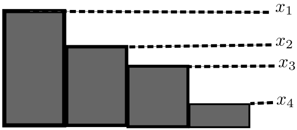
A chain distribution on the ground set X is a distribution over $S \subseteq X$ who's support forms a chain in the inclusion order.

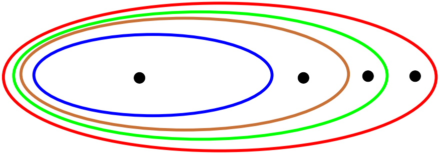


Chain Distributions

Chain Distribution with Given Marginals

Fix the ground set $X = \{1, \ldots, n\}$. The chain distribution with marginals $x \in [0,1]^n$ is the unique chain distribution $D^{\mathcal{L}}(x)$ satisfying $\mathbf{Pr}_{S \sim D^{\mathcal{L}}(x)}[i \in S] = x_i$ for all $i \in X$.





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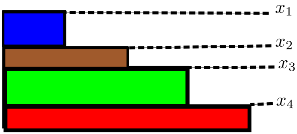

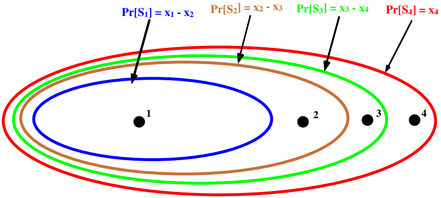

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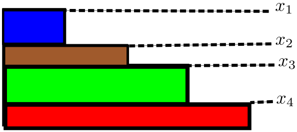

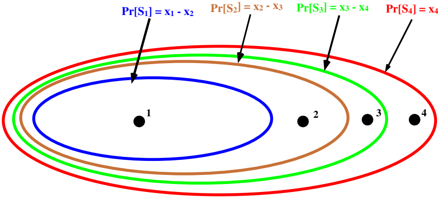

$D^{\mathcal{L}}(x)$ is the distribution given by the following process:

- Sort $x_1 \geq x_2 \ldots \geq x_n$
- Let $S_i = \{1, \ldots, i\}$
- Let $\mathbf{Pr}[S_i] = x_i - x_{i+1}$

The Lovasz Extension

Definition

The Lovasz extension of a set function f is defined as follows.

$$f^{\mathcal{L}}(x) = \mathop{\mathbf{E}}_{S \sim D^{\mathcal{L}}(x)} f(S)$$

i.e. the Lovasz extension at x is the expected value of a set drawn from the unique chain distribution with marginals x.

Observations

- $f^{\mathcal{L}}$ is an extension, since the chain distribution with marginals $y \in \{0,1\}^n$ is the point distribution at y.

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Observations

- $f^{\mathcal{L}}$ is an extension, since the chain distribution with marginals $y \in \{0,1\}^n$ is the point distribution at y.
- $f^{\mathcal{L}}(x)$ is the expected value of f on some distribution on $\{0,1\}^n$ with marginals x. Since $f^{-}(x)$ chooses the "lowest" such distribution, we have $f^{\mathcal{L}}(x) \geq f^{-}(x)$.

Equivalence of the Convex Closure and Lovasz Extension

Theorem

If f is submodular, then $f^{\mathcal{L}} = f^{-}$.

Converse holds: if f not submodular, then $f^{\mathcal{L}}$ not convex. (won't prove)

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Intuition

- Recall: $f^-(x)$ evaluates f on the "lowest" distribution with marginals x
- It turns out that, when f is submodular, this lowest distribution is the chain distribution $D^{\mathcal{L}}(x)$.

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Intuition

- Recall: $f^-(x)$ evaluates f on the "lowest" distribution with marginals x
- It turns out that, when f is submodular, this lowest distribution is the chain distribution $D^{\mathcal{L}}(x)$.
- Contingent on marginals x, submodularity implies that cost is minimized by "packing" as many elements together as possible
    - diminishing marginal returns
- This gives the chain distribution

It suffices to show that the chain distribution with marginals x is in fact the “lowest” distribution with marginals x.

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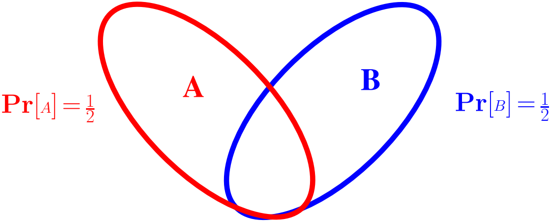
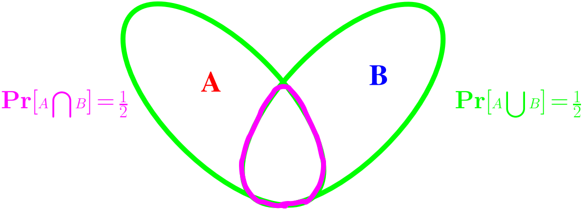


$\frac{1}{2} f(A) + \frac{1}{2} f(B)$

It suffices to show that the chain distribution with marginals x is in fact the “lowest” distribution with marginals x.

Proof (Special case)

- Take a distribution $\mathcal{D}$ on two “crossing” sets A and B, with probability 0.5 each.
- Consider “uncrossing” A and B, replacing them with $A \bigcap B$ and $A \bigcup B$, with probability 0.5 each.
  - Yields a chain distribution supported on $A \bigcap B$ and $A \bigcup B$.
  - Marginals don’t change
  - By submodularity, expected value can only go down.



$$\frac{1}{2}f(A) + \frac{1}{2}f(B) \geq \frac{1}{2}f(A \bigcap B) + \frac{1}{2}f(A \bigcup B)$$

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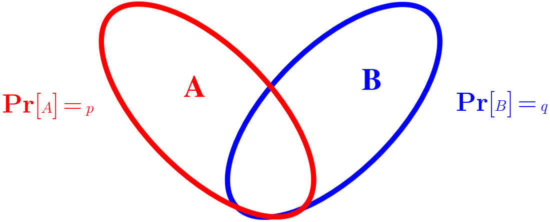
- Take a distribution $\mathcal{D}$ on two "crossing" sets A and B, with probabilities $p \leq q$.



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- Take a distribution $\mathcal{D}$ on two "crossing" sets A and B, with probabilities $p \leq q$.
- Consider "uncrossing" a probability mass of p from each of A,B.
  - Yields a chain distribution supported on $A \bigcap B$, B, and $A \bigcup B$.
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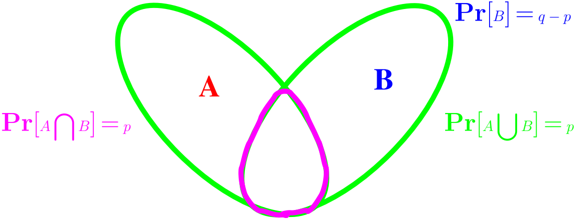

$$pf(A) + qf(B) \geq pf(A \bigcap B) + pf(A \bigcup B) + (q-p)f(B)$$

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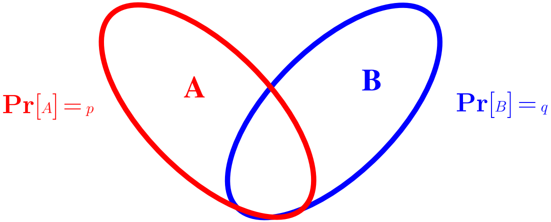
- Take a distribution $\mathcal{D}$ which includes two “crossing” sets A and B in its support, with probabilities $p \leq q$.



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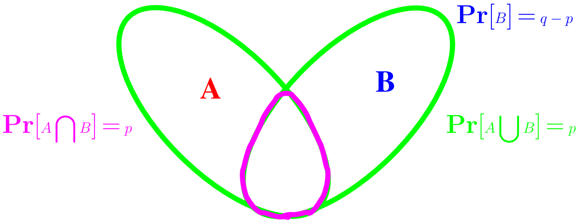

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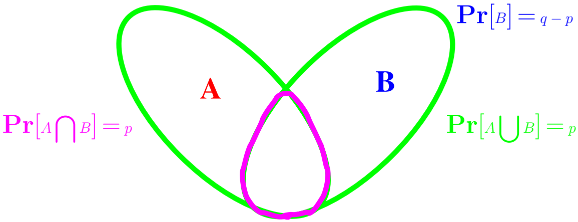
- Consider “uncrossing” a probability mass of p from each of A, B.
  - Marginals don’t change
  - By submodularity, expected value can only go down.
- Makes $\mathcal{D}$ “closer” to being a chain distribution
  - The bounded potential function $\mathbf{E}_{S \sim \mathcal{D}}[|S|^2]$ increases



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Minimizing the Lovasz Extension

Because $f^{\mathcal{L}} = f^{-}$, we know the following:

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Therefore, minimizing f reduces to the following convex optimization problem

Minimizing the Lovasz Extension

$$\begin{array}{ll} \text{minimize} & f^{\mathcal{L}}(x) \\ \text{subject to} & x \in [0,1]^n \end{array}$$

Recall: Solvability of Convex Optimization

Weak Solvability

An algorithm weakly solves our optimization problem if it takes in approximation parameter $\epsilon > 0$, runs in $\mathrm{poly}(n, \log \frac{1}{\epsilon})$ time, and returns $x \in [0,1]^n$ which is ϵ-optimal:

$$f^{\mathcal{L}}(x) \leq \min_{y \in [0,1]^n} f^{\mathcal{L}}(y) + \epsilon[\max_{y \in [0,1]^n} f^{\mathcal{L}}(y) - \min_{y \in [0,1]^n} f^{\mathcal{L}}(y)]$$

Recall: Solvability of Convex Optimization

Polynomial Solvability of CP

In order to weakly minimize $f^{\mathcal{L}}$, we need the following operations to run in $\mathrm{poly}(n)$ time:

1. Compute a starting ellipsoid $E \supseteq [0,1]^n$ with $\frac{\mathbf{vol}(E)}{\mathbf{vol}([0,1]^n)} = O(\exp(n))$.
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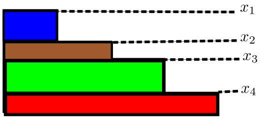
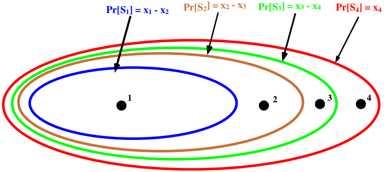
1 and 2 are trivial.

First order Oracle for $f^{\mathcal{L}}$





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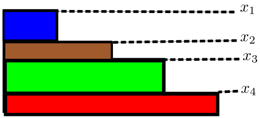

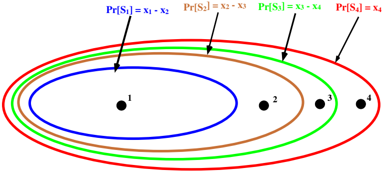

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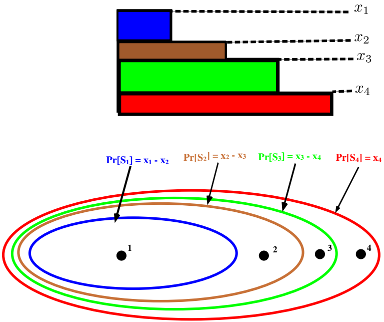

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- $f^{\mathcal{L}}$ is peicewise linear, so can compute a sub-gradient.

Recovering an Optimal Set

We can get an ϵ-optimal solution x^* to the optimization problem in $\mathrm{poly}(n, \log \frac{1}{\epsilon})$ time.

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- We can identify this set by examining the chain distribution with marginals x^*

Outline

1. Introduction to Submodular Functions

2. Unconstrained Submodular Minimization
   - Definition and Examples
   - The Convex Closure and the Lovasz Extension
   - Wrapping up

3. Monotone Submodular Maximization s.t. a Matroid Constraint
   - Definition and Examples
   - Warmup: Cardinality Constraint
   - General Matroid Constraints

Recall: Optimizing Submodular Functions

	Maximization	Minimization
Unconstrained	NP-hard $\frac{1}{2}$ approximation	Polynomial time via convex opt
Constrained	Usually NP-hard $1 - 1/e$ (mono, matroid) $O(1)$ (“nice” constraints)	Usually NP-hard to apx. Few easy special cases

Recall: Optimizing Submodular Functions

	Maximization	Minimization
Unconstrained	NP-hard $\frac{1}{2}$ approximation	Polynomial time via convex opt
Constrained	Usually NP-hard $1 - 1/e$ (mono, matroid) $O(1)$ ("nice" constraints)	Usually NP-hard to apx. Few easy special cases

Problem Definition

Given a non-decreasing and normalized submodular function $f : 2^X \to \mathbb{R}_+$ on a finite ground set X, and a matroid $M = (X, \mathcal{I})$

$$\begin{array}{ll} \text{maximize} & f(S) \\ \text{subject to} & S \in \mathcal{I} \end{array}$$

- Non-decreasing: $f(S) \leq f(T)$ for $S \subseteq T$
- Normalized: $f(\emptyset) = 0$.

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Representation

As before, we work in the value oracle and independence oracle models. Namely, we assume we have access to a subroutine evaluating $f(S)$, and a subroutine for checking whether $S \in \mathcal{I}$, each in constant time.

Examples

Maximum Coverage

X is the left hand side of a graph, and $f(S)$ is the total number of neighbors of S.

- Can think of $i \in X$ as a set, and $f(S)$ as the total “coverage” of S.

Goal is to cover as much of the RHS as possible with k LHS nodes.

Social Influence

- X is the family of nodes in a social network
- A meme, idea, or product is adopted at a set of nodes S
- $f(S)$ is the expected number of nodes in the network which end up adopting the idea.
- Goal is to obtain maximum influence subject to a constraint
  - Cardinality
  - Transversal
  - …

Combinatorial Allocation

- G is a set of goods
- $f_i(B)$ is submodular utility of agent $i \in N$ for bundle $B \subseteq G$
- Allocation: A partition $(B_1, \ldots, B_n)$ of G among agents.
- Aggregate utility is $\sum_i f_i(B_i)$.

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- Aggregate utility is $\sum_i f_i(B_i)$.
- Let $X = G \times N$ be the set of good/agent pairs
- Allocations correspond to subsets S of X in which at most one "copy" of each good is chosen
    - Partition matroid constraint
- $f(S) = \sum_{i \in N} f_i(\{j \in G : (j, i) \in S\})$
    - Submodular

Complexity

Theorem

Maximizing a submodular function subject to a matroid constraint is NP-hard, and NP-hard to approximate to within any better than a factor of $1 - 1/e$.

- Holds even for max coverage subject to a cardinality constraint (Feige ’98)

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Goal

An algorithm in the value oracle and independence oracle models which

- Runs in time $\mathrm{poly}(n)$
- Returns a feasible set $S^* \in \mathcal{I}$ satisfying $f(S^*) \geq (1 - 1/e) \max_{S \in \mathcal{I}} f(S)$.

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Holds for arbitrary matroid, but much simpler for uniform matroids.

Subject to a Cardinality Constraint

Problem Definition

Given a non-decreasing and normalized submodular function $f : 2^X \to \mathbb{R}_+$ on a finite ground set X with $|X| = n$, and an integer $k \leq n$

$$\begin{array}{ll} \text{maximize} & f(S) \\ \text{subject to} & |S| \leq k \end{array}$$

- k-uniform matroid constraint

The Greedy Algorithm

The following is the straightforward adaptation of the greedy algorithm for maximizing modular functions over a matroid.

The Greedy Algorithm

```
1. $S \leftarrow \emptyset$
2. While $|S| < k$
     - Choose $e \in X$ maximizing $f(S \bigcup \{e\})$
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Theorem

The greedy algorithm is a $(1 - 1/e)$ *approximation algorithm for maximizing a monotone, normalized, and submodular function subject to a cardinality constraint.*

Contraction/Conditioning

Let $f : 2^X \to \mathbb{R}$ and $A \subseteq X$. Define $f_A(S) = f(A \bigcup S) - f(A)$.

Lemma

If f is monotone and submodular, then f_A is monotone, submodular, and normalized for any A.

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  - $f_A(S) = f(S \cup A) - f(A) \leq f(T \cup A) - f(A) = f_A(T)$.
- Submodular:

$$\begin{aligned} f_A(S) + f_A(T) &= f(S \cup A) - f(A) + f(T \cup A) - f(A) \\ &\geq f(S \cup T \cup A) - f(A) + f((S \cap T) \cup A) - f(A) \\ &= f_A(S \cup T) + f_A(S \cap T) \end{aligned}$$

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- Therefore, $\max_{j \in A} f(\{j\}) \geq \frac{1}{|A|} f(A)$

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- We will show that the suboptimality $OPT - f(S)$ shrinks by a factor of $(1 - 1/k)$ each iteration
- After k iterations, it has shrunk to $(1 - 1/k)^k \leq 1/e$ from its original value

$$OPT - f(S) \leq \frac{1}{e} OPT$$

$$(1 - 1/e) OPT \leq f(S)$$

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- Therefore, suboptimality decreases by factor of $1 - \frac{1}{k}$, as needed.

From Uniform to Arbitrary Matroid

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- The discrete greedy algorithm is now only a $1/2$ approximation
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  - $f(a) = f(b) = 1$, $f(c) = f(ac) = 1 + \epsilon$, $f(ab) = f(bc) = f(abc) = 2$

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- Nevertheless, a continuous greedy algorithm gives $1 - 1/e$
- Approach resembles that for minimization
  - Define a continous extension of f
  - Optimize continuous extension over matroid polytope
  - Extract an integer point

The Multilinear Extension

Multilinear Extension

Given a set function $f : \{0,1\}^n \to \mathbb{R}$, its multilinear extension $F : [0,1]^n \to \mathbb{R}$ evaluated at $x \in [0,1]^n$ gives the expected value of $f(S)$ for the random set S which includes each i independently with probability x_i.

$$F(x) = \sum_{S \subseteq X} f(S) \prod_{i \in S} x_i \prod_{i \neq S} (1 - x_i)$$

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- For each point x, evaluates f on the independent distribution $D(x)$
- Clearly an extension of f
- Not concave (or convex) in general
  - Recall f with $f(\emptyset) = 0$ and $f(\{1\}) = f(\{2\}) = f(\{1,2\}) = 1$
  - $F(x) = 1 - (1 - x_1)(1 - x_2)$

Easy Properties of the Multilinear Extension

Normalized

When f is normalized, $F(0) = 0$

Follows from the fact that F is an extension of f

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Nondecreasing

When f is monotone non-decreasing, $F(x) \leq F(y)$ whenever $x \preceq y$ component-wise.

Increasing the probability of selecting each element increases the expected value.

Up-concavity

Even though F is not concave, it is concave in “upwards” directions.

Up-concavity

Assume f is submodular. For every $\vec{a} \in [0,1]^n$ and $\vec{d} \in [0,1]^n$ satisfying $d \succeq 0$, the function $g(t) = F(\vec{a} + \vec{d}\, t)$ is a concave function of $t \in \mathbb{R}$.

Proof Sketch

- By multivariate chain rule: $\frac{d^2 g}{dt^2} = d^T (\bigtriangledown^2 F) d$
- The Hessian $\bigtriangledown^2 F$ is not negative semi-definite, so can’t conclude that g is concave for arbitrary directions d
- Multilinearity implies second partial derivatives $\frac{\partial^2 F}{\partial x_i^2}$ are zero
- Submodularity implies mixed derivatives $\frac{\partial^2 F}{\partial x_i \partial x_j}$ are nonpositive
    - Diminishing marginal returns + coupling argument
- Therefore $\frac{d^2 g}{dt^2} = d^T (\bigtriangledown^2 F) d \leq 0$ for $\vec{d} \succeq 0$

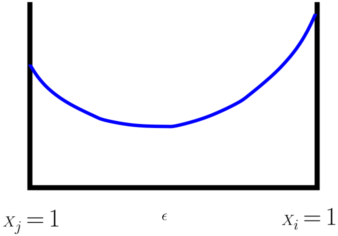
Cross-convexity

Nevertheless, F is convex in "cross" directions.

Cross-convexity

Assume f is submodular. For every $a \in [0,1]^n$ and $\vec{d} = e_i - e_j$ for some $i, j \in X$, the function $g(t) = F(\vec{a} + \vec{d}\, t)$ is a convex function of $t \in \mathbb{R}$.

- Trading off one item's probability for another's gives convex curve
- Follows from submodularity: as we "remove" j, the marginal benefit of "adding" i increases



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Proof

- $\frac{d^2 g}{dt^2} = d^T(\bigtriangledown^2 F)d = \frac{\partial^2 F}{\partial x_i^2} + \frac{\partial^2 F}{\partial x_j^2} - 2\frac{\partial^2 F}{\partial x_i \partial x_j}$
- By multilinearity, $\frac{\partial^2 F}{\partial x_i^2} = \frac{\partial^2 F}{\partial x_j^2} = 0$
- We already argued that submodularity implies $\frac{\partial^2 F}{\partial x_i \partial x_j} \leq 0$

Algorithm Outline

Step A: Continuous Greedy Algorithm

Computes a $1 - 1/e$ approximation to the following continuous (non-convex) optimization problem.

$$\begin{array}{ll} \text{maximize} & F(x) \\ \text{subject to} & x \in \mathcal{P}(\mathcal{M}) \end{array}$$

- i.e. Computes x^* s.t. $F(x^*) \geq (1 - 1/e) \max \{F(x) : x \in \mathcal{P}(\mathcal{M})\}$

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No! $D(x^*)$ may be largely supported on infeasible sets (i.e. not independent in matroid $\mathcal{M}$).

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Step B: Pipage Rounding

"Rounds" x^* to some vertex y^* of the matroid polytope (i.e. an independent set) satisfying

$$f(y^*) = F(y^*) \geq F(x^*)$$

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- A-priori, not obvious that such a y^* exists

Step A: Continuous Greedy Algorithm

- Feasible polytope $\mathcal{P} \subseteq [0,1]^n$
  - Downwards Closed: If $y \in \mathcal{P}$ and $\vec{0} \preceq x \preceq y$ then $x \in \mathcal{P}$ also.
- Objective function $F : [0,1]^n \to \mathbb{R}_+$ which is non-decreasing, up-concave, and normalized ($F(\vec{0}) = 0$).

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  - Position at time t given by $x(t)$.

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- Discretized to time steps of ϵ, which we will assume to be arbitrarily small for convenience of analysis, but may be taken to be $1/\operatorname{poly}(n)$ in the actual implementation.

Step A: Continuous Greedy Algorithm

Continuous Greedy Algorithm (F,$\mathcal{P}$, ϵ)

```
1. $x(0) \leftarrow \vec{0}$
2. For $t \in [0, \epsilon, 2\epsilon, \ldots, 1 - \epsilon]$
       Let $y(t) \in \mathrm{argmax}_{y \in \mathcal{P}} \{\bigtriangledown F(x(t)) \cdot y\}$
       $x(t + \epsilon) \leftarrow x(t) + \epsilon y(t)$
3. Return $x(1)$
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- I.e. When the particle is at x, it moves in direction y maximizing the linear function $\triangledown F(x) \cdot y$ over $y \in \mathcal{P}$
    - The direction is actually a vertex of our matroid polytope
    - This is NOT gradient ascent

Step A: Continuous Greedy Algorithm

Continuous Greedy Algorithm (F,$\mathcal{P}$, ϵ)

```
1. $x(0) \leftarrow \vec{0}$
2. For $t \in [0, \epsilon, 2\epsilon, \ldots, 1 - \epsilon]$
       Let $y(t) \in \text{argmax}_{y \in \mathcal{P}} \{\bigtriangledown F(x(t)) \cdot y\}$
       $x(t + \epsilon) \leftarrow x(t) + \epsilon y(t)$
3. Return $x(1)$
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- I.e. When the particle is at x, it moves in direction y maximizing the linear function $\bigtriangledown F(x) \cdot y$ over $y \in \mathcal{P}$
    - The direction is actually a vertex of our matroid polytope
    - This is NOT gradient ascent
- Observe: Algorithm forms a convex combination of $\frac{1}{\epsilon}$ vertices of the polytope $\mathcal{P}$, each with weight ϵ.
    - $x(1) \in \mathcal{P}$.

Theorem

In the limit as $\epsilon \to 0$, the continuous greedy algorithm outputs a $1 - 1/e$ approximation to maximizing $F(x)$ over $\mathcal{P}$.

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Proof Sketch

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&\geq F(\max(x^*, x(t))) - F(x(t)) \\
&\geq OPT - F(x(t))
\end{aligned}$$

Theorem

In the limit as $\epsilon \to 0$, the continuous greedy algorithm outputs a $1 - 1/e$ approximation to maximizing $F(x)$ over $\mathcal{P}$.

Proof Sketch

- $v(t) = F(x(t))$ satisfies $\frac{dv}{dt} \geq OPT - v$.
- Differential equation $\frac{dv}{dt} = OPT - v$ with boundary condition $v(0) = 0$ has a unique solution

$$v(t) = OPT(1 - e^{-t})$$

- $v(1) \geq OPT(1 - 1/e)$

Implementation Details

Continuous Greedy Algorithm (F,$\mathcal{P}$, ϵ)

```
1. $x(0) \leftarrow \vec{0}$
2. For $t \in [0, \epsilon, 2\epsilon, \ldots, 1-\epsilon]$
     - Let $y(t) \in \operatorname{argmax}_{y \in \mathcal{P}} \{\bigtriangledown F(x(t)) \cdot y\}$
     - $x(t+\epsilon) \leftarrow x(t) + \epsilon y(t)$
3. Return $x(1)$
```

When F is multilinear extension of submodular f, and $\mathcal{P} = \mathcal{P}(\mathcal{M})$ for matroid $\mathcal{M}$.

- $\bigtriangledown F(x)$ is not readily available, but can be estimated "accurately enough" using $\mathrm{poly}(n)$ random samples from $D(x)$, w.h.p.
- Step 2 can be implemented because $\mathcal{P}$ is solvable
- Discretization: Taking $\epsilon = 1/O(n^2)$ is "fine enough"
- Both the above introduce error into the approximation guarantee, yielding $1 - 1/e - 1/O(n)$ w.h.p
- This can be shaved off to $1 - 1/e$ with some additional "tricks".

- The following algorithm takes x in matroid base polytope $\mathcal{P}_{base}(\mathcal{M})$, and non-decreasing cross-convex function F, and outputs integral y with $F(y) \geq F(x)$

PipageRounding ($\mathcal{M}$,x, F)

While x contains a fractional entry

1. Let T be a minimum-size tight set containing a fractional entry
    - i.e. $x(T) = rank_{\mathcal{M}}(T)$, $i \in T$ for some i with $x_i \in (0,1)$, and $|T|$ is as small as possible.
2. Let $j \in T$ be such that $j \neq i$ and x_j is fractional.
3. Let $x(\mu) = x + \mu(e_i - e_j)$, and maximize $F(x(\mu))$ subject to $x(\mu) \in \mathcal{P}(\mathcal{M})$.
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Theorem

On input $x \in \mathcal{P}_{base}(\mathcal{M})$, Pipage rounding terminates in $O(n^2)$ iterations, and outputs a matroid vertex y with $f(y) = F(y) \geq F(x)$.

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Step 1

- T is a subset of every other tight set containing i, because tight sets form a lattice
    - A lattice is a family of sets closed under intersection and union.
- Proof:
    - Tight sets are the minimizers of the set function $rank_{\mathcal{M}}(S) - x(S)$
    - This set function is submodular.
    - Minimizers of a submodular function form a lattice.

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Step 2

- Since rank is integer valued, any tight set containing fractional variable should have another.

PipageRounding ($\mathcal{M}$,x, F)

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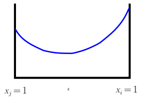
Step 3+4

- Either the number of fractional variables decreases, or a smaller tight set containing x_i or x_j is created.
    - Why smaller? T remains tight, and if R is a new tight set then by lattice property so is $T \bigcap R$
- Therefore this terminates in $O(n^2)$ iterations
- $F(x)$ does not decrease by definition of step 3



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- Efficient implementation of continuous greedy algorithm follows from matroid optimization and basic concentration bounds
- Efficient implementation of each iteration of Pipage rounding will be on HW

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Theorem

The continuous greedy algorithm followed by Pipage rounding gives a $(1 - 1/e)$ approximation algorithm for maximizing a monotone, normalized, and submodular function subject to a matroid constraint.