

CS675: Convex and Combinatorial Optimization  
Spring 2018  
Introduction to Linear Programming

Instructor: Shaddin Dughmi

# Outline

- 1 Linear Programming Basics
- 2 Duality and Its Interpretations
- 3 Properties of Duals
- 4 Weak and Strong Duality
- 5 Formal Proof of Strong Duality of LP
- 6 Consequences of Duality
- 7 More Examples of Duality

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# A Brief History

- The forefather of convex optimization problems, and the most ubiquitous.
- Developed by Kantorovich during World War II (1939) for planning the Soviet army's expenditures and returns. Kept secret.
- Discovered a few years later by George Dantzig, who in 1947 developed the simplex method for solving linear programs
- John von Neumann developed LP duality in 1947, and applied it to game theory
- Polynomial-time algorithms: Ellipsoid method (Khachiyan 1979), interior point methods (Karmarkar 1984).

$$\begin{array}{ll} \text{minimize (or maximize)} & c^\top x \\ \text{subject to} & a_i^\top x \leq b_i, \quad \text{for } i \in \mathcal{C}^1. \\ & a_i^\top x \geq b_i, \quad \text{for } i \in \mathcal{C}^2. \\ & a_i^\top x = b_i, \quad \text{for } i \in \mathcal{C}^3. \end{array}$$

- Decision variables:  $x \in \mathbb{R}^n$
- Parameters:
  - $c \in \mathbb{R}^n$  defines the linear objective function
  - $a_i \in \mathbb{R}^n$  and  $b_i \in \mathbb{R}$  define the  $i$ 'th constraint.

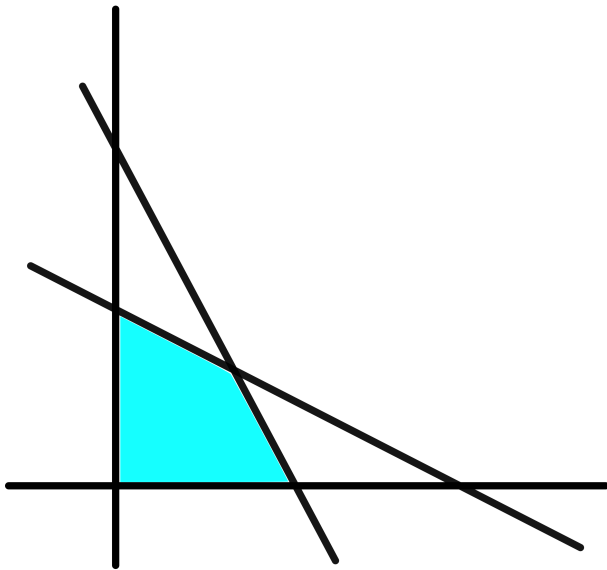
# Standard Form

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & a_i^\top x \leq b_i, \quad \text{for } i = 1, \dots, m. \\ & x_j \geq 0, \quad \text{for } j = 1, \dots, n. \end{array}$$

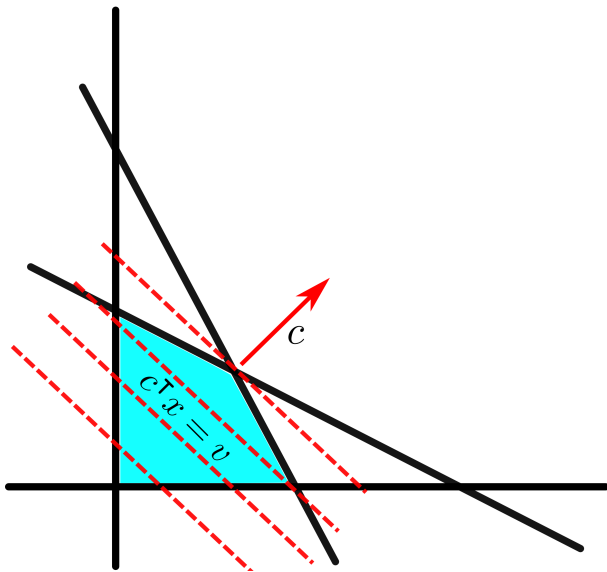
Every LP can be transformed to this form

- minimizing  $c^\top x$  is equivalent to maximizing  $-c^\top x$
- $\geq$  constraints can be flipped by multiplying by  $-1$
- Each equality constraint can be replaced by two inequalities
- Unconstrained variable  $x_j$  can be replaced by  $x_j^+ - x_j^-$ , where both  $x_j^+$  and  $x_j^-$  are constrained to be nonnegative.

# Geometric View



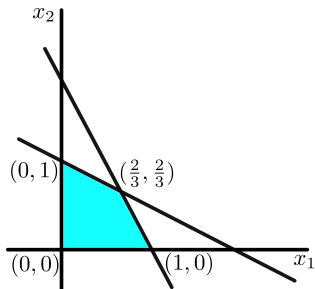
# Geometric View





# A 2-D example

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 2 \\ & 2x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$



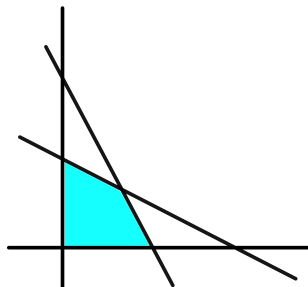
# Application: Optimal Production

- $n$  products,  $m$  raw materials
- Every unit of product  $j$  uses  $a_{ij}$  units of raw material  $i$
- There are  $b_i$  units of material  $i$  available
- Product  $j$  yields profit  $c_j$  per unit
- Facility wants to maximize profit subject to available raw materials

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# Terminology

- **Hyperplane**: The region defined by a linear equality
- **Halfspace**: The region defined by a linear inequality  $a_i^T x \leq b_i$ .
- **Polyhedron**: The intersection of a set of linear inequalities
  - Feasible region of an LP is a polyhedron
- **Polytope**: Bounded polyhedron
  - Equivalently: **convex hull** of a finite set of points
- **Vertex**: A point  $x$  is a vertex of polyhedron  $P$  if  $\nexists y \neq 0$  with  $x + y \in P$  and  $x - y \in P$
- **Face** of  $P$ : The intersection with  $P$  of a hyperplane  $H$  disjoint from the interior of  $P$



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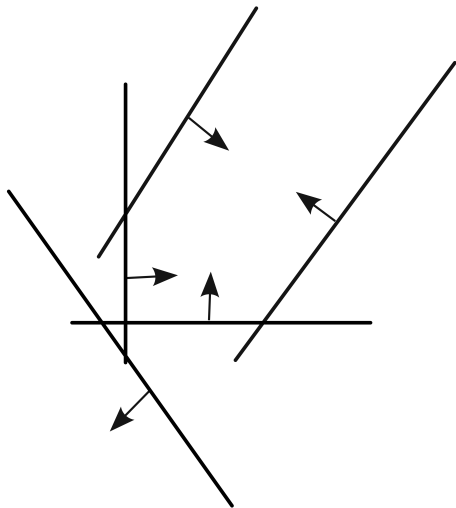
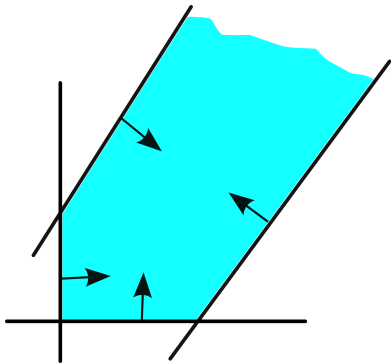
## Fact

At a vertex,  $n$  linearly independent constraints are satisfied with equality (a.k.a. **tight**)

# Basic Facts about LPs and Polyhedrons

## Fact

An LP either has an optimal solution, or is **unbounded** or **infeasible**



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If an LP in standard form has an optimal solution, then it has a vertex optimal solution.



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- There is  $y \neq 0$  s.t.  $x \pm y$  are feasible
- $y$  is perpendicular to the objective function and the tight constraints at  $x$ .
  - i.e.  $c^T y = 0$ , and  $a_i^T y = 0$  whenever the  $i$ 'th constraint is tight for  $x$ .

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- Can choose  $y$  s.t.  $y_j < 0$  for some  $j$
- Let  $\alpha$  be the largest constant such that  $x + \alpha y$  is feasible
  - Such an  $\alpha$  exists
- An additional constraint becomes tight at  $x + \alpha y$ , a contradiction.

## Corollary

If an LP in standard form has an optimal solution, then there is an optimal solution with at most  $m$  non-zero variables.

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & a_i^\top x \leq b_i, \quad \text{for } i = 1, \dots, m. \\ & x_j \geq 0, \quad \text{for } j = 1, \dots, n. \end{array}$$

- e.g. for optimal production with  $n$  products and  $m$  raw materials, there is an optimal plan with at most  $m$  products.

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# Linear Programming Duality

## Primal LP

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

## Dual LP

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y = c \\ & y \geq 0 \end{array}$$

- $A \in \mathbb{R}^{m \times n}$ ,  $c \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$
- $y_i$  is the **dual variable** corresponding to primal constraint  $A_i x \leq b_i$
- $A_j^T y \geq c_j$  is the **dual constraint** corresponding to primal variable  $x_j$

# Linear Programming Duality: Standard Form, and Visualization

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# Interpretation 1: Economic Interpretation

Recall the Optimal Production problem from last lecture

- $n$  products,  $m$  raw materials
- Every unit of product  $j$  uses  $a_{ij}$  units of raw material  $i$
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- Dual variable  $y_i$  is a proposed **price** per unit of raw material  $i$
- Dual price vector is feasible if facility has incentive to sell materials
- Buyer wants to spend as little as possible to buy materials

## Interpretation 2: Finding the Best Upperbound

Consider the simple LP from last lecture

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 2 \\ & 2x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

- We found that the optimal solution was at  $(\frac{2}{3}, \frac{2}{3})$ , with an optimal value of  $4/3$ .

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- We found that the optimal solution was at  $(\frac{2}{3}, \frac{2}{3})$ , with an optimal value of  $4/3$ .
- What if, instead of finding the optimal solution, we sought to find an upperbound on its value by combining inequalities?
  - Each inequality implies an upper bound of 2
  - Multiplying each by  $\frac{1}{3}$  and summing gives  $x_1 + x_2 \leq 4/3$ .

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$$y^T Ax \leq y^T b$$

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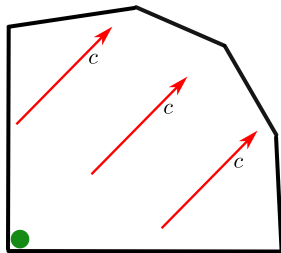
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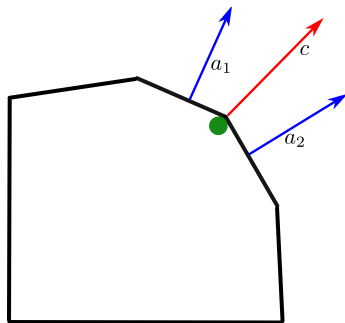
- The dual LP can be thought of as trying to find the best upperbound on the primal that can be achieved this way.

## Interpretation 3: Physical Forces



- Apply force field  $c$  to a ball inside bounded polytope  $Ax \leq b$ .

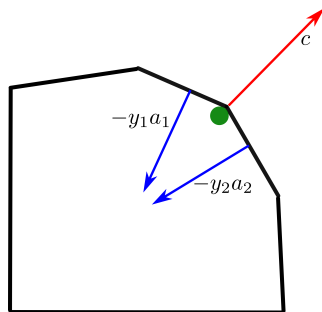
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- Eventually, ball will come to rest against the walls of the polytope.

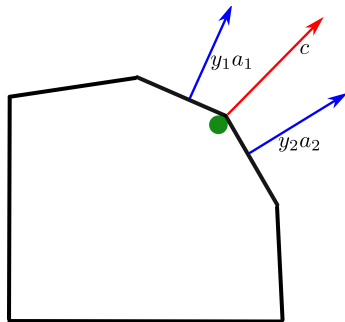


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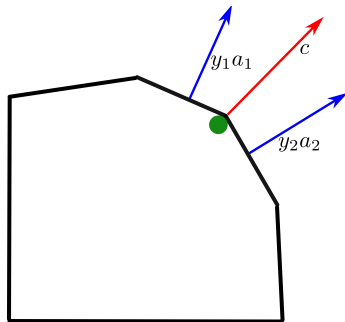
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- Since the ball is still,  $c^T = \sum_i y_i a_i = y^T A$ .
- Dual can be thought of as trying to minimize “work”  $\sum_i y_i b_i$  to bring ball back to origin by moving polytope
- We will see that, at optimality, only the walls adjacent to the ball push (Complementary Slackness)

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# Duality is an Inversion

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## Dual LP

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c \\ & y \geq 0 \end{array}$$

## Duality is an Inversion

Given a primal LP in standard form, the dual of its dual is itself.

# Correspondance Between Variables and Constraints

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$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{s.t.} & \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\ & x_j \geq 0, \quad \text{for } j \in [n]. \end{array}$$

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- The  $i$ 'th primal constraint gives rise to the  $i$ 'th dual variable  $y_i$

# Correspondance Between Variables and Constraints

## Primal LP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{s.t.} & \\ \mathbf{y}_i : & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\ & x_j \geq 0, \quad \text{for } j \in [n]. \end{array}$$

## Dual LP

$$\begin{array}{ll} \min & \sum_{i=1}^m b_i y_i \\ \text{s.t.} & \\ \mathbf{x}_j : & \sum_{i=1}^m a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\ & y_i \geq 0, \quad \text{for } i \in [m]. \end{array}$$

- The  $i$ 'th primal constraint gives rise to the  $i$ 'th dual variable  $y_i$
- The  $j$ 'th primal variable  $x_j$  gives rise to the  $j$ 'th dual constraint



# Syntactic Rules

## Primal LP

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & \\ y_i : \quad & a_i x \leq b_i, \quad \text{for } i \in \mathcal{C}_1. \\ y_i : \quad & a_i x = b_i, \quad \text{for } i \in \mathcal{C}_2. \\ & x_j \geq 0, \quad \text{for } j \in \mathcal{D}_1. \\ & x_j \in \mathbb{R}, \quad \text{for } j \in \mathcal{D}_2. \end{aligned}$$

## Dual LP

$$\begin{aligned} \min \quad & b^\top y \\ \text{s.t.} \quad & \\ x_j : \quad & \bar{a}_j^\top y \geq c_j, \quad \text{for } j \in \mathcal{D}_1. \\ x_j : \quad & \bar{a}_j^\top y = c_j, \quad \text{for } j \in \mathcal{D}_2. \\ & y_i \geq 0, \quad \text{for } i \in \mathcal{C}_1. \\ & y_i \in \mathbb{R}, \quad \text{for } i \in \mathcal{C}_2. \end{aligned}$$

## Rules of Thumb

- Loose constraint (i.e. inequality)  $\Rightarrow$  tight dual variable (i.e. nonnegative)
- Tight constraint (i.e. equality)  $\Rightarrow$  loose dual variable (i.e. unconstrained)

# Outline

- 1 Linear Programming Basics
- 2 Duality and Its Interpretations
- 3 Properties of Duals
- 4 Weak and Strong Duality**
- 5 Formal Proof of Strong Duality of LP
- 6 Consequences of Duality
- 7 More Examples of Duality

# Weak Duality

## Primal LP

$$\begin{aligned} &\text{maximize} && c^\top x \\ &\text{subject to} && Ax \leq b \\ & && x \geq 0 \end{aligned}$$

## Dual LP

$$\begin{aligned} &\text{minimize} && b^\top y \\ &\text{subject to} && A^\top y \geq c \\ & && y \geq 0 \end{aligned}$$

## Theorem (Weak Duality)

*For every primal feasible  $x$  and dual feasible  $y$ , we have  $c^\top x \leq b^\top y$ .*

## Corollary

- *If primal and dual both feasible and bounded,  $OPT(\text{Primal}) \leq OPT(\text{Dual})$*
- *If primal is unbounded, dual is infeasible*
- *If dual is unbounded, primal is infeasible*

# Weak Duality

## Primal LP

$$\begin{aligned} &\text{maximize} && c^\top x \\ &\text{subject to} && Ax \leq b \\ &&& x \geq 0 \end{aligned}$$

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## Theorem (Weak Duality)

*For every primal feasible  $x$  and dual feasible  $y$ , we have  $c^\top x \leq b^\top y$ .*

## Corollary

*If  $x$  is primal feasible, and  $y$  is dual feasible, and  $c^\top x = b^\top y$ , then both are optimal.*

## Economic Interpretation

If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

# Interpretation of Weak Duality

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If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

## Upperbound Interpretation

Self explanatory

# Interpretation of Weak Duality

## Economic Interpretation

If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

## Upperbound Interpretation

Self explanatory

## Physical Interpretation

Work required to bring ball back to origin by pulling polytope is at least potential energy difference between origin and primal optimum.

# Proof of Weak Duality

## Primal LP

maximize  $c^T x$   
subject to  $Ax \leq b$   
 $x \geq 0$

## Dual LP

minimize  $b^T y$   
subject to  $A^T y \geq c$   
 $y \geq 0$

$$c^T x \leq y^T Ax \leq y^T b$$



# Strong Duality

## Primal LP

maximize  $c^T x$   
subject to  $Ax \leq b$   
 $x \geq 0$

## Dual LP

minimize  $b^T y$   
subject to  $A^T y \geq c$   
 $y \geq 0$

## Theorem (Strong Duality)

*If either the primal or dual is feasible and bounded, then so is the other and  $OPT(Primal) = OPT(Dual)$ .*

# Interpretation of Strong Duality

## Economic Interpretation

Buyer can offer prices for raw materials that would make facility indifferent between production and sale.

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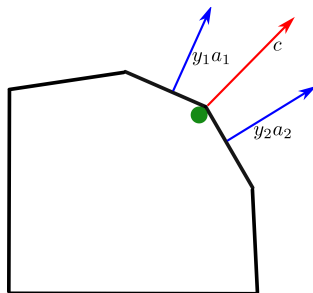
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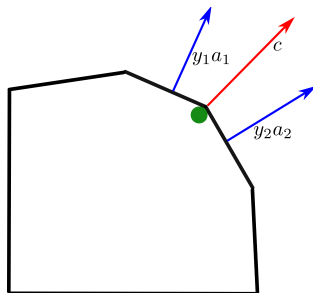
There is an assignment of forces to the walls of the polytope that brings ball back to the origin without wasting energy.

# Informal Proof of Strong Duality



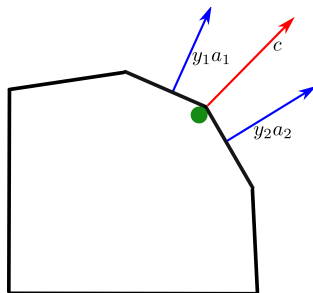
- Recall the physical interpretation of duality

# Informal Proof of Strong Duality



- Recall the physical interpretation of duality
- When ball is stationary at  $x$ , we expect force  $c$  to be neutralized only by constraints that are tight. i.e. force multipliers  $y \geq 0$  s.t.
  - $y^T A = c$
  - $y_i (b_i - a_i x) = 0$

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  - $y^T A = c$
  - $y_i(b_i - a_i x) = 0$

$$y^T b - c^T x = y^T b - y^T A x = \sum_i y_i(b_i - a_i x) = 0$$

We found a primal and dual solution that are equal in value!

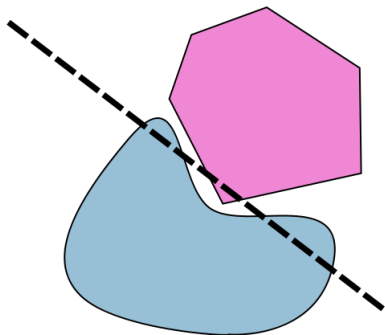
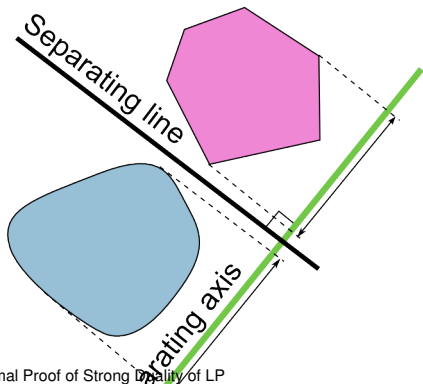
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## Separating Hyperplane Theorem

If  $A, B \subseteq \mathbb{R}^n$  are disjoint convex sets, then there is a hyperplane separating them. That is, there is  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}$  such that  $a^\top x \leq b$  for every  $x \in A$  and  $a^\top y \geq b$  for every  $y \in B$ . Moreover, if both  $A$  and  $B$  are closed and at least one of them is compact, then there is a hyperplane strictly separating them (i.e.  $a^\top x < b$  for  $x \in A$  and  $a^\top y > b$  for  $y \in B$ ).



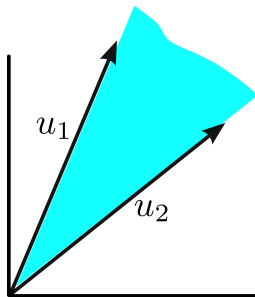
## Definition

A **convex cone** is a convex subset of  $\mathbb{R}^n$  which is closed under nonnegative scaling and convex combinations.

## Definition

The convex cone **generated** by vectors  $u_1, \dots, u_m \in \mathbb{R}^n$  is the set of all nonnegative-weighted sums of these vectors (also known as **conic combinations**).

$$\text{Cone}(u_1, \dots, u_m) = \left\{ \sum_{i=1}^m \alpha_i u_i : \alpha_i \geq 0 \forall i \right\}$$

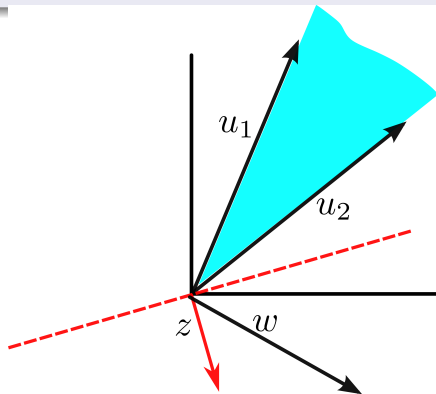


The following follows from the separating hyperplane Theorem (try to prove it).

## Farkas' Lemma

Let  $\mathcal{C}$  be the **convex cone** generated by vectors  $u_1, \dots, u_m \in \mathbb{R}^n$ , and let  $w \in \mathbb{R}^n$ . Exactly one of the following is true:

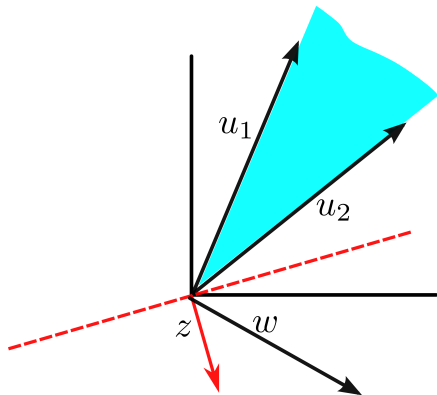
- $w \in \mathcal{C}$
- There is  $z \in \mathbb{R}^n$  such that  $z \cdot u_i \leq 0$  for all  $i$ , and  $z \cdot w > 0$ .



## Equivalently: Theorem of the Alternative

Exactly one of the following is true for  $U = [u_1, \dots, u_m]$

- The system  $Uy = w, y \geq 0$  has a solution
- The system  $U^T z \leq 0, z^T w > 0$  has a solution.



# Formal Proof of Strong Duality

## Primal LP

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & Ax \leq b \end{array}$$

## Dual LP

$$\begin{array}{ll} \text{minimize} & b^\top y \\ \text{subject to} & A^\top y = c \\ & y \geq 0 \end{array}$$

Given  $v \in \mathbb{R}$ , by Farkas' Lemma exactly one of the following is true

- 1 The system  $\begin{pmatrix} A^\top & 0 \\ b^\top & 1 \end{pmatrix} w = \begin{pmatrix} c \\ v \end{pmatrix}$ ,  $w \geq 0$  has a solution.
  - Let  $y \in \mathbb{R}_+^m$  and  $\delta \in \mathbb{R}_+$  be such that  $w = \begin{pmatrix} y \\ \delta \end{pmatrix}$
  - Implies dual is feasible and  $OPT(dual) \leq v$
- 2 The system  $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} z \leq 0$ ,  $z^\top \begin{pmatrix} c \\ v \end{pmatrix} > 0$  has a solution.
  - Let  $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ , where  $z_1 \in \mathbb{R}^n$  and  $z_2 \in \mathbb{R}$  with  $z_2 \leq 0$
  - When  $z_2 \neq 0$ ,  $x = -z_1/z_2$  is feasible and  $c^\top x > v$
  - When  $z_2 = 0$ , primal is either infeasible or unbounded, and dual is infeasible (prove it)

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# Complementary Slackness

## Primal LP

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- Let  $s_i = (b - Ax)_i$  be the  $i$ 'th **primal slack variable**
- Let  $t_j = (A^T y - c)_j$  be the  $j$ 'th **dual slack variable**



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## Complementary Slackness

$x$  and  $y$  are optimal if and only if

- $x_j t_j = 0$  for all  $j = 1, \dots, n$
- $y_i s_i = 0$  for all  $i = 1, \dots, m$

	$x_1$	$x_2$	$x_3$	$x_4$	
$y_1$	$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$	$b_1$
$y_2$	$a_{21}$	$a_{22}$	$a_{23}$	$a_{24}$	$b_2$
$y_3$	$a_{31}$	$a_{32}$	$a_{33}$	$a_{34}$	$b_3$
	$c_1$	$c_2$	$c_3$	$c_4$	

## Economic Interpretation

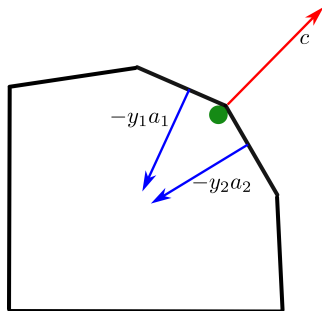
Given an optimal primal production vector  $x$  and optimal dual offer prices  $y$ ,

- Facility produces only products for which it is indifferent between sale and production.
- Only raw materials that are binding constraints on production are priced greater than 0

# Interpretation of Complementary Slackness

## Physical Interpretation

Only walls adjacent to the balls equilibrium position push back on it.



# Proof of Complementary Slackness

## Primal LP

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

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Gap between primal and dual objectives is 0 if and only if complementary slackness holds.

# Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.



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  - Exactly  $n$  dual constraints are loose

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  - Exactly  $n$  dual constraints are loose
- $n$  loose dual constraints impose  $n$  tight primal constraints
  - Assuming non-degeneracy, solving the linear equation yields a unique primal optimum solution  $x$ .

# Sensitivity Analysis

## Primal LP

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

## Dual LP

$$\begin{array}{ll} \text{minimize} & y^\top b \\ \text{subject to} & A^\top y \geq c \\ & y \geq 0 \end{array}$$

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Sometimes, we want to examine how the optimal value of our LP changes with its parameters  $c$  and  $b$

## Sensitivity Analysis

Let  $OPT = OPT(A, c, b)$  be the optimal value of the above LP. Let  $x$  and  $y$  be the primal and dual optima.

- $\frac{\partial OPT}{\partial c_j} = x_j$  when  $x$  is the unique primal optimum.
- $\frac{\partial OPT}{\partial b_i} = y_i$  when  $y$  is the unique dual optimum.

# Sensitivity Analysis

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Sometimes, we want to examine how the optimal value of our LP changes with its parameters  $c$  and  $b$

## Economic Interpretation of Sensitivity Analysis

- A small increase  $\delta$  in  $c_j$  increases profit by  $\delta \cdot x_j$
- A small increase  $\delta$  in  $b_i$  increases profit by  $\delta \cdot y_i$ 
  - $y_i$  measures the “marginal value” of resource  $i$  for production

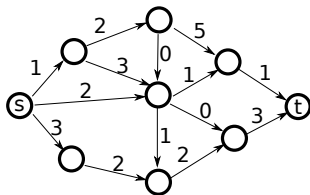


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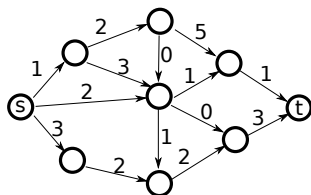
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# Shortest Path

Given a directed network  $G = (V, E)$  where edge  $e$  has length  $\ell_e \in \mathbb{R}_+$ , find the minimum cost path from  $s$  to  $t$ .



# Shortest Path



## Primal LP

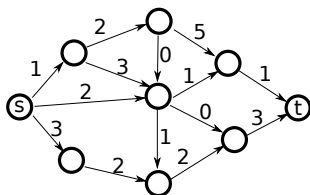
$$\begin{aligned} \min \quad & \sum_{e \in E} \ell_e x_e \\ \text{s.t.} \quad & \sum_{e \rightarrow v} x_e - \sum_{v \rightarrow e} x_e = \delta_v, \quad \forall v \in V. \\ & x_e \geq 0, \quad \forall e \in E. \end{aligned}$$

## Dual LP

$$\begin{aligned} \max \quad & y_t - y_s \\ \text{s.t.} \quad & y_v - y_u \leq \ell_e, \quad \forall (u, v) \in E. \end{aligned}$$

Where  $\delta_v = -1$  if  $v = s$ ,  $1$  if  $v = t$ , and  $0$  otherwise.

# Shortest Path



## Primal LP

$$\begin{aligned} \min \quad & \sum_{e \in E} \ell_e x_e \\ \text{s.t.} \quad & \sum_{e \rightarrow v} x_e - \sum_{v \rightarrow e} x_e = \delta_v, \quad \forall v \in V. \\ & x_e \geq 0, \quad \forall e \in E. \end{aligned}$$

## Dual LP

$$\begin{aligned} \max \quad & y_t - y_s \\ \text{s.t.} \quad & y_v - y_u \leq \ell_e, \quad \forall (u, v) \in E. \end{aligned}$$

Where  $\delta_v = -1$  if  $v = s$ ,  $1$  if  $v = t$ , and  $0$  otherwise.

## Interpretation of Dual

Stretch  $s$  and  $t$  as far apart as possible, subject to edge lengths.

# Maximum Weighted Bipartite Matching

Set  $B$  of buyers, and set  $G$  of goods. Buyer  $i$  has value  $w_{ij}$  for good  $j$ , and interested in at most one good. Find maximum value assignment of goods to buyers.

# Maximum Weighted Bipartite Matching

## Primal LP

$$\begin{aligned} \max \quad & \sum_{i,j} w_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j \in G} x_{ij} \leq 1, \quad \forall i \in B. \\ & \sum_{i \in B} x_{ij} \leq 1, \quad \forall j \in G. \\ & x_{ij} \geq 0, \quad \forall i \in B, j \in G. \end{aligned}$$

## Dual LP

$$\begin{aligned} \min \quad & \sum_{i \in B} u_i + \sum_{j \in G} p_j \\ \text{s.t.} \quad & u_i + p_j \geq w_{ij}, \quad \forall i \in B, j \in G. \\ & u_i \geq 0, \quad \forall i \in B. \\ & p_j \geq 0, \quad \forall j \in G. \end{aligned}$$

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## Interpretation of Dual

- $p_j$  is price of good  $j$
- $u_i$  is utility of buyer  $i$
- Complementary Slackness:
  - A buyer  $i$  only grabs goods  $j$  maximizing  $w_{ij} - p_j$
  - Only fully assigned goods have non-zero price
  - A buyer with nonzero utility must receive an item

# 2-Player Zero-Sum Games

## Rock-Paper-Scissors

	$R$	$P$	$S$
$R$	0	1	-1
$P$	-1	0	1
$S$	1	-1	0

- Two players, row and column
- Game described by matrix  $A$
- When row player plays pure strategy  $i$  and column player plays pure strategy  $j$ , row player pays column player  $A_{ij}$



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- **Mixed Strategy**: distribution over pure strategies
- Assume players know each other's mixed strategies but not coin flips

## 2-Player Zero-Sum Games

- Assume row player moves first with distribution  $y \in \Delta_m$ 
  - Loss as a function of column's strategy given by  $y^\top A$
  - A best response by column is pure strategy  $j$  maximizing  $(y^\top A)_j$

	$x_1$	$x_2$	$x_3$	$x_4$
$y_1$	$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$
$y_2$	$a_{21}$	$a_{22}$	$a_{23}$	$a_{24}$
$y_3$	$a_{31}$	$a_{32}$	$a_{33}$	$a_{34}$

## 2-Player Zero-Sum Games

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### Row Moves First

$$\begin{array}{ll} \min & \max_j (y^T A)_j \\ \text{s.t.} & \sum_{i=1}^m y_i = 1 \\ & y \geq \vec{0} \end{array}$$

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### Row Moves First

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$$\begin{array}{ll} \max & v \\ \text{s.t.} & v\vec{1} - Ax \leq \vec{0} \\ & \sum_{j=1}^n x_j = 1 \\ & x \geq \vec{0} \end{array}$$

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These two optimization problems are LP Duals!

## Weak Duality

- $u^* \geq v^*$
- Zero sum games have a second mover advantage (weakly)



# Duality and Zero Sum Games

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## Strong Duality (Minimax Theorem)

- $u^* = v^*$
- There is no second or first mover advantage in zero sum games with mixed strategies
- Each player can guarantee  $u^* = v^*$  regardless of other's strategy.
- $y^*, x^*$  are simultaneously best responses to each other (Nash Equilibrium)

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## Complementary Slackness

$x^*$  randomizes over pure best responses to  $y^*$ , and vice versa.

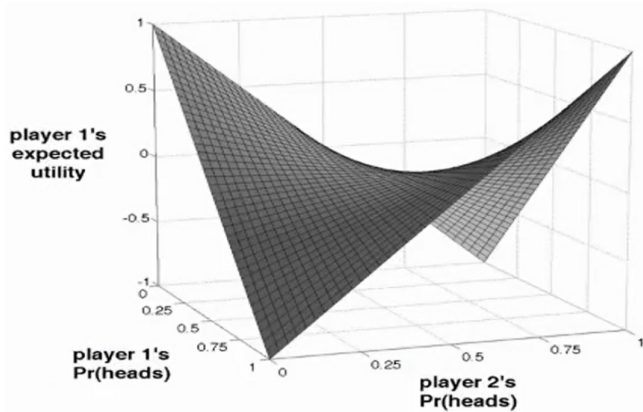
# Saddle Point Interpretation

Consider the matching pennies game

	$H$	$T$
$H$	$-1$	$1$
$T$	$1$	$-1$

- Unique equilibrium: each player randomizes uniformly
- If row player deviates, he pays out more
- If column player deviates, he gets paid less

# Saddle Point Interpretation



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