# CS675: Convex and Combinatorial Optimization
# Spring 2018
# Combinatorial Problems as Linear and Convex Programs

Instructor: Shaddin Dughmi

# Outline

# Combinatorial Vs Convex Optimization

- In CS, discrete problems are traditionally viewed/analyzed using discrete mathematics and combinatorics
  - Algorithms are combinatorial in nature (greedy, dynamic programming, divide and conquor, etc)

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  - Usually linear programs, but increasingly more general convex programs

# Combinatorial Vs Convex Optimization

- In CS, discrete problems are traditionally viewed/analyzed using discrete mathematics and combinatorics
  - Algorithms are combinatorial in nature (greedy, dynamic programming, divide and conquor, etc)
- In OR and optimization community, these problems are often expressed as continuous optimization problems
  - Usually linear programs, but increasingly more general convex programs
- Increasingly in recent history, it is becoming clear that combining both viewpoints is the way to go
  - Better algorithms (runtime, approximation)
  - Structural insights (e.g. market clearing prices in matching markets)
  - Unifying theories and general results (Matroids, submodular optimization, constraint satisfaction)

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# Discrete Problems as Linear Programs

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  - Dantzig's original application was the problem of matching 70 people to 70 jobs!
- This is not surprising, since almost any finite family of discrete objects can be encoded as a finite subset of Euclidean space
  - Convex hull of that set is a polytope
  - E.g. spanning trees, paths, cuts, TSP tours, assignments...

# Discrete Problems as Linear Programs

- LP algorithms typically require representation as a “small” family of inequalities,
  - Not possible in general (Say when problem is NP-hard, assuming $(P \neq NP)$)
  - Shown unconditionally impossible in some cases (e.g. TSP)

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- But, in many cases, polyhedra in inequality form can be shown to encode a combinatorial problems at the vertices

**Next**

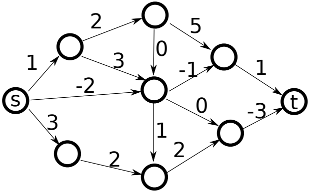
We examine some combinatorial problems through the lense of LP and convex optimization, starting with shortest path.

# Outline

# The Shortest Path Problem

Given a directed graph $G = (V, E)$ with cost $c_e \in \mathbb{R}$ on edge $e$, find the minimum cost path from $s$ to $t$.

- We use $n$ and $m$ to denote $|V|$ and $|E|$, respectively.
- We allow costs to be negative, but assume no negative cycles



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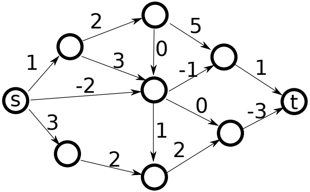


When costs are nonnegative, Dijkstra's algorithm finds the shortest path from $s$ to every other node in time $O(m + n \log n)$.

Using primal/dual paradigm, we will design a polynomial-time algorithm that works when graph has negative edges but no negative cycles

# Note: Negative Edges and Complexity

- When the graph has no negative cycles, there is a shortest path which is simple
- When the graph has negative cycles, there may not be a shortest path from $s$ to $t$.
- In these cases, the algorithm we design can be modified to “fail gracefully” by detecting such a cycle
  - Can be used to detect arbitrage opportunities in currency exchange networks

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- In these cases, the algorithm we design can be modified to “fail gracefully” by detecting such a cycle
  - Can be used to detect arbitrage opportunities in currency exchange networks
- In the presence of negative cycles, finding the shortest simple path is NP-hard (by reduction from Hamiltonian cycle)

# An LP Relaxation of Shortest Path

Consider the following LP

## Primal Shortest Path LP

$$\begin{aligned}
&\text{min } \textstyle\sum_{e \in E} c_e x_e \\
&\text{s.t.} \\
&\sum_{e \to v} x_e - \sum_{v \to e} x_e = \delta_v, \quad && \forall v \in V. \\
&x_e \geq 0, && \forall e \in E.
\end{aligned}$$

where $\delta_v = -1$ if $v = s$, 1 if $v = t$, and $0$ otherwise.

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  - Indicator vector $x_P$ of $s - t$ path $P$ is a feasible solution, with cost as given by the objective
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- This is a relaxation of the shortest path problem
  - Indicator vector $x_P$ of $s - t$ path $P$ is a feasible solution, with cost as given by the objective
  - Fractional feasible solutions may not correspond to paths
- A-priori, it is conceivable that optimal value of LP is less than length of shortest path.

# Integrality of the Shortest Path Polyhedron

$$\begin{aligned}
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\end{aligned}$$

We will show that above LP encodes the shortest path problem exactly

## Claim

When $c$ satisfies the no-negative-cycles property, the indicator vector of the shortest $s - t$ path is an optimal solution to the LP.

# Dual LP

We will use the following LP dual

**Primal LP**

min $\sum_{e \in E} c_e x_e$
s.t.

$$\sum_{e \to v} x_e - \sum_{v \to e} x_e = \delta_v, \quad \forall v \in V.$$
$$x_e \geq 0, \quad \forall e \in E.$$

**Dual LP**

max $y_t - y_s$
s.t.

$$y_v - y_u \leq c_e, \quad \forall (u,v) \in E.$$

- Interpretation of dual variables $y_v$: “height” or “potential”
- Relative potential of vertices constrained by length of edge between them (triangle inequality)
- Dual is trying to maximize relative potential of $s$ and $t$,

# Proof Using the Dual

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- Let $x^*$ be indicator vector of shortest s-t path
  - Feasible for primal
- Let $y^*_v$ be shortest path distance from $s$ to $v$
  - Feasible for dual (by triangle inequality)
- $\sum_e c_e x^*_e = y^*_t - y^*_s$, so both $x^*$ and $y^*$ optimal.

# Integrality of Polyhedra

A stronger statement is true:

**Integrality of Shortest Path LP**

The vertices of the polyhedral feasible region are precisely the indicator vectors of simple paths in $G$.

- Implies that there always exists an optimal solution which is a path whenever LP is bounded and feasible
- Reduces computing shortest path in graphs with no negative cycles to finding optimal vertex of LP

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2. Fact: For every LP vertex $x$ there is objective $c$ such that $x$ is unique optimal. (Prove it!)
3. Since such a $c$ satisfies no-negative-cycles property, claim on previous slide shows that $x$ is integral.

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In general, the approach we took applies in many contexts: To show a polytope's vertices integral, it suffices to show that there is an integral optimal for any objective which admits an optimal solution.

# Outline

# Ford's Algorithm

## Primal LP

min $\sum_{e \in E} c_e x_e$
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## Dual LP

max $y_t - y_s$
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$$y_v - y_u \leq c_e, \quad \forall e = (u, v) \in E.$$

For convenience, add $(s, v)$ of length $\infty$ when one doesn't exist.

## Ford's Algorithm

1. $y_v = c_{(s,v)}$ and $pred(v) = s$ for $v \neq s$
2. $y_s = 0$, $pred(s) = null$.
3. While some dual constraint is violated, i.e. $y_v > y_u + c_e$ for some $e = (u, v)$
   - Set $pred(v) = u$ (To get from $s$ to $v$, take shortcut through $u$)
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4. Output the path $t, pred(t), pred(pred(t)), \ldots, s$.

# Correctness

## Lemma (Loop Invariant 1)

Assuming no negative cycles, $pred$ defines a path $P$ from $s$ to $t$, of length at most $y_t - y_s$. (Hence also $y_t - y_s \geq distance(s,t)$)

## Interpretation

- Ford's algorithm maintains an (initially infeasible) dual $y$
- Also maintains feasible primal $P$ of length $\leq$ dual objective $y_t - y_s$
- Iteratively "fixes" dual $y$, tending towards feasibility
- Once $y$ is feasible, weak duality implies $P$ optimal.

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If Ford's algorithm terminates, then it outputs a shortest path from $s$ to $t$

Algorithms of this form, that output a matching primal and dual solution, are called Primal-Dual Algorithms.

# Termination

**Lemma (Loop Invariant 2)**

Assuming no negative cycles, $y_v$ is the length of some simple path from $s$ to $v$.

# Termination

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## Theorem (Termination)

When the graph has no negative cycles, Ford's algorithm terminates in a finite number of steps.

## Proof

- The graph has a finite number $N$ of simple paths
- By loop invariant 2, every dual variable $y_v$ is the length of some simple path.
- Dual variables are nonincreasing throughout algorithm, and one decreases each iteration.
- There can be at most $nN$ iterations.

# Observation: Single sink shortest paths

## Ford's Algorithm

1. $y_v = c_{(s,v)}$ and $pred(v) = s$ for $v \neq s$
2. $y_s = 0$, $pred(s) = null$.
3. While some dual constraint is violated, i.e. $y_v > y_u + c_e$ for some $e = (u,v)$
   - Set $pred(v) = u$ (To get from $s$ to $v$, take shortcut through $u$)
   - Set $y_v = y_u + c_e$
4. Output the path $t, pred(t), pred(pred(t)), \ldots, s$.

## Observation

Algorithm does not depend on $t$ till very last step. So essentially solves the single-source shortest path problem. i.e. finds shortest paths from $s$ to all other vertices $v$.

# Loop Invariant 1

We prove Loop Invariant 1 through two Lemmas

## Lemma (Loop Invariant 1a)

For every node $w$, we have $y_w - y_{pred(w)} \geq c_{pred(w),w}$

## Proof

- Fix $w$
- Holds at first iteration
- Preserved by Induction on iterations
  - If neither $y_w$ nor $y_{pred(w)}$ updated, nothing changes.
  - If $y_w$ (and $pred(w)$) updated, then $y_w = y_{pred(w)} + c_{pred(w),w}$
  - $y_{pred(w)}$ updated, it only goes down, preserving inequality.

# Loop Invariant 1

## Lemma (Invariant 1b)

Assuming no negative cycles, pred forms a directed tree rooted out of $s$.

We denote this path from $s$ to a node $w$ by $P(s, w)$.

## Proof

- Holds at first iteration
- For a contradiction, consider iteration of first violation
  - $v$ and $u$ with $y_v > y_u + c_{u,v}$
- $P(s, u)$ passes through $v$
  - Otherwise tree property preserved by setting $pred(v) = u$
- Let $P(v, u)$ be the portion of $P(s, u)$ starting at $v$.
- By Invariant 1a, and telescoping sum, length of $P(v, u)$ is at most $y_u - y_v$.
- Length of cycle $\{P(v, u), (u, v)\}$ at most $y_u - y_v + c_{u,v} < 0$.

# Summarizing Loop Invariant 1

**Lemma (Invariant 1a)**

For every node $w$, we have $y_w - y_{pred(w)} \geq c_{pred(w),w}$.

- By telescoping sum, can bound $y_w - y_s$ when pred leads back to $s$

**Lemma (Invariant 1b)**

Assuming no negative cycles, pred forms a directed tree rooted out of $s$.

- Implies that $y_s$ remains $0$

**Corollary (Loop Invariant 1)**

Assuming no negative cycles, $pred$ defines a path $P(s, w)$ from $s$ to each node $w$, of length at most $y_w - y_s = y_w$. (Hence $y_w \geq distance(s, w)$)

# Loop Invariant 2

**Lemma (Loop Invariant 2)**

Assuming no negative cycles, $y_w$ is the length of some simple path $Q(s, w)$ from $s$ to $w$, for all $w$.

Proof is technical, by induction, so we will skip. Instead, we will modify Ford's algorithm to guarantee polynomial time termination.

# Bellman-Ford Algorithm

The following algorithm fixes an (arbitrary) order on edges $E$

## Bellman-Ford Algorithm

1. $y_v = c_{(s,v)}$ and $pred(v) = s$ for $v \neq s$
2. $y_s = 0$, $pred(s) = null$.
3. While y is infeasible for the dual
   - For $e = (u, v)$ in order, if $y_v > y_u + c_e$ then
     - Set $pred(v) = u$ (To get from $s$ to $v$, take shortcut through $u$)
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4. Output the path $t, pred(t), pred(pred(t)), \ldots, s$.

## Note

Correctness follows from the correctness of Ford's Algorithm.

# Runtime

**Theorem**

*Bellman-Ford terminates after $n - 1$ scans through $E$, for a total runtime of $O(nm)$.*

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*Bellman-Ford terminates after $n - 1$ scans through $E$, for a total runtime of $O(nm)$.*

Follows immediately from the following Lemma

**Lemma**

After $k$ scans through $E$, vertices $v$ with a shortest $s - v$ path consisting of $\leq k$ edges are correctly labeled. (i.e., $y_v = distance(s, v)$)

# Proof

## Lemma

After $k$ scans through $E$, vertices $v$ with a shortest $s - v$ path consisting of $\leq k$ edges are correctly labeled. (i.e., $y_v = distance(s, v)$)

## Proof

- Holds for $k = 0$
- By induction on $k$.
  - Assume it holds for $k - 1$.
  - Let $v$ be a node with a shortest path $P$ from $s$ with $k$ edges.
  - $P = \{Q, e\}$, for some $e = (u, v)$ and $s - u$ path $Q$, where $Q$ is a shortest $s - u$ path and $Q$ has $k - 1$ edges.
  - By inductive hypothesis, $u$ is correctly labeled before $e$ is scanned for $k$th time – i.e. $y_u = distance(s, u)$.
  - Therefore, $v$ is correctly labeled $y_v = y_u + c_{u,v} = distance(s, v)$ after $e$ is scanned for $k$th time

# A Note on Negative Cycles

**Question**

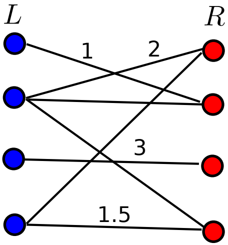
What if there are negative cycles? What does that say about LP? What about Ford's algorithm?

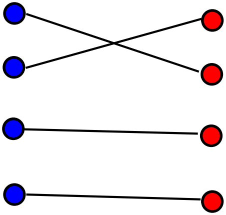
# Outline

# The Max-Weight Bipartite Matching Problem

Given a bipartite graph $G = (V, E)$, with $V = L \bigcup R$, and weights $w_e$ on edges $e$, find a maximum weight matching.

- Matching: a set of edges covering each node at most once
- We use $n$ and $m$ to denote $|V|$ and $|E|$, respectively.
- Equivalent to maximum weight / minimum cost perfect matching.



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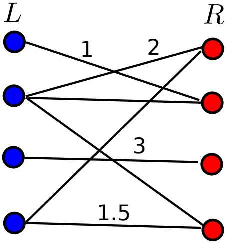


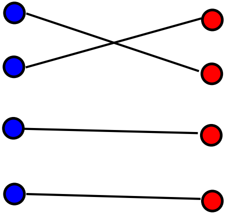

Our focus will be less on algorithms, and more on using polyhedral interpretation to gain insights about a combinatorial problem.

# An LP Relaxation of Bipartite Matching

## Bipartite Matching LP

max $\sum_{e \in E} w_e x_e$

s.t.

$$\sum_{e \in \delta(v)} x_e \leq 1, \qquad \forall v \in V.$$

$$x_e \geq 0, \qquad \forall e \in E.$$

# An LP Relaxation of Bipartite Matching

**Bipartite Matching LP**

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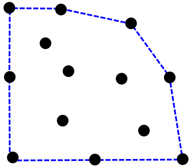
\end{aligned}
$$

- Feasible region is a polytope $\mathcal{P}$ (i.e. a bounded polyhedron)
- This is a relaxation of the bipartite matching problem
  - Integer points in $\mathcal{P}$ are the indicator vectors of matchings.

$$\mathcal{P} \cap \mathbb{Z}^m = \{x_M : M \text{ is a matching}\}$$

# Integrality of the Bipartite Matching Polytope

$$\sum_{e \in \delta(v)} x_e \le 1, \quad \forall v \in V.$$
$$x_e \ge 0, \quad \forall e \in E.$$



**Theorem**

The feasible region of the matching LP is the convex hull of indicator vectors of matchings.

$$\mathcal{P} = \text{convexhull}\,\{x_M : M \text{ is a matching}\}$$

# Integrality of the Bipartite Matching Polytope

$$\sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V.$$
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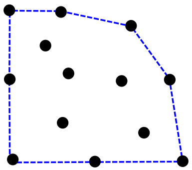

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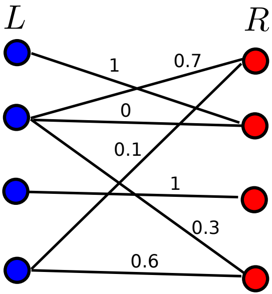
## Note

- This is the strongest guarantee you could hope for of an LP relaxation of a combinatorial problem
- Solving LP is equivalent to solving the combinatorial problem
- Stronger guarantee than shortest path LP from last time

# Proof

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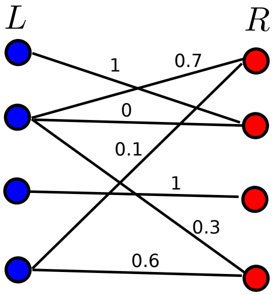


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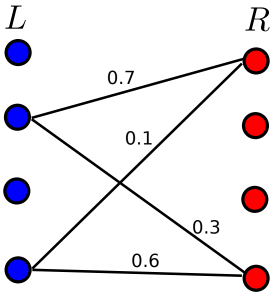


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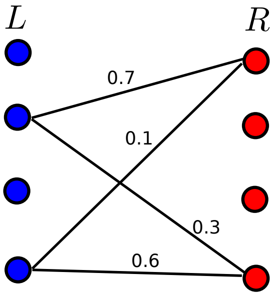


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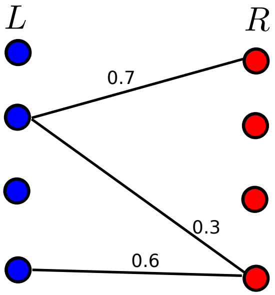


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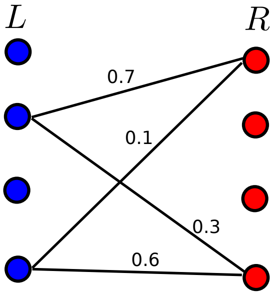


## Case 1: Cycle $C$

- Let $C = (e_1, \ldots, e_k)$, with $k$ even
- There is $\epsilon > 0$ such that adding $\pm\epsilon(+1, -1, \ldots, +1, -1)$ to $x_C$ preserves feasibility
- $x$ is the midpoint of $x + \epsilon(+1, -1, ..., +1, -1)_C$ and $x - \epsilon(+1, -1, \ldots, +1, -1)_C$, so $x$ is not a vertex.

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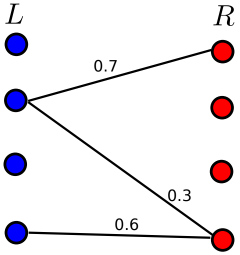


## Case 2: Maximal Path $P$

- Let $P = (e_1, \ldots, e_k)$, going through vertices $v_0, v_1, \ldots, v_k$
- By maximality, $e_1$ is the only edge of $v_0$ with non-zero $x$-weight
  - Similarly for $e_k$ and $v_k$.
- There is $\epsilon > 0$ such that adding $\pm\epsilon(+1, -1, \ldots, ?1)$ to $x_P$ preserves feasibility
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# Related Fact: Birkhoff Von-Neumann Theorem

$$\sum_{e \in \delta(v)} x_e = 1, \quad \forall v \in V.$$
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- The analogous statement holds for the perfect matching LP above, by an essentially identical proof.

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The set of $n \times n$ doubly stochastic matrices is the convex hull of $n \times n$ permutation matrices.

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By Caratheodory's theorem, we can express every doubly stochastic matrix as a convex combination of $n^2 + 1$ permutation matrices.

We will see later: this decomposition can be computed efficiently!

# Outline

# Total Unimodularity

We could have proved integrality of the bipartite matching LP using a more general tool

**Definition**

A matrix A is Totally Unimodular if every square submatrix has determinant $0$, $+1$ or $-1$.

**Theorem**

*If $A \in \mathbb{R}^{m \times n}$ is totally unimodular, and $b$ is an integer vector, then $\{x : Ax \leq b, x \geq 0\}$ has integer vertices.*

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## Proof

- Non-zero entries of vertex $x$ are solution of $A'x' = b'$ for some nonsingular square submatrix $A'$ and corresponding sub-vector $b'$
- Cramer's rule:

$$x'_i = \frac{\det(A'_i | b')}{\det A'}$$

# Total Unimodularity of Bipartite Matching

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- If all columns of $A'$ have two $1$'s,
  - Partition rows (vertices) into $L$ and $R$
  - Sum of rows $L$ is $(1, 1, \ldots, 1)$, similarly for $R$
  - $A'$ is singular, so $\det A' = 0$.

# Outline

# Primal and Dual LPs

**Primal LP**

$$\max \sum_{e\in E} w_e x_e$$
s.t.
$$\sum_{e\in\delta(v)} x_e \le 1, \qquad \forall v \in V.$$
$$x_e \ge 0, \qquad \forall e \in E.$$

**Dual LP**

$$\min \sum_{v\in V} y_v$$
s.t.
$$y_u + y_v \ge w_e, \quad \forall e = (u,v) \in E.$$
$$y_v \succeq 0, \qquad \forall v \in V.$$

- Primal interpretation: Player 1 looking to build a set of projects
  - Each edge $e$ is a project generating “profit” $w_e$
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- Dual interpretation: Player 2 looking to buy resources
  - Offer a price $y_v$ for each resource.
  - Prices should incentivize player 1 to sell resources
  - Want to pay as little as possible.

# Vertex Cover Interpretation

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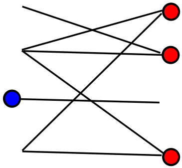
When edge weights are $1$, binary solutions to dual are vertex covers

**Definition**

$C \subseteq V$ is a vertex cover if every $e \in E$ has at least one endpoint in $C$



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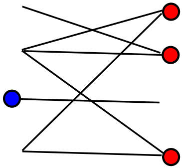

- Dual is a relaxation of the minimum vertex cover problem for bipartite graphs.
- By weak duality: min-vertex-cover $\geq$ max-cardinality-matching

# König's Theorem

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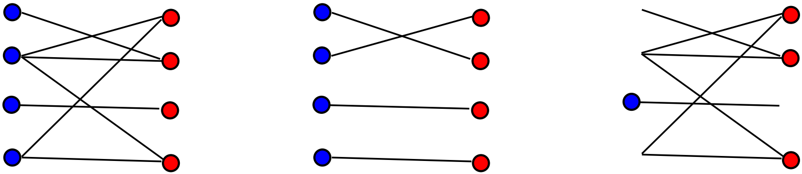
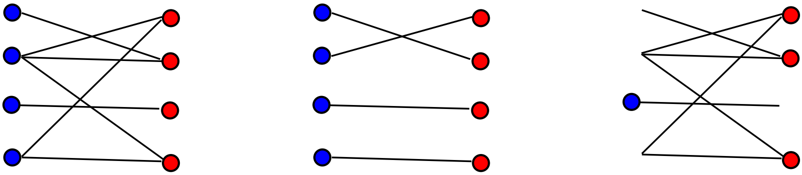
## König's Theorem

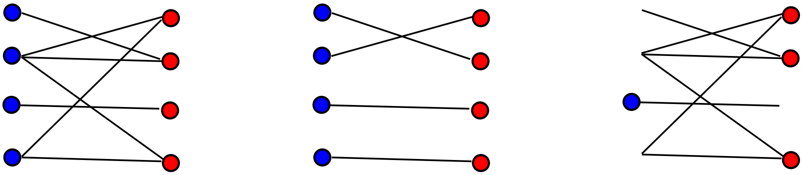
In a bipartite graph, the cardinality of the maximum matching is equal to the cardinality of the minimum vertex cover.

i.e. the dual LP has an optimal integral solution





- Let $M(G)$ be a max cardinality of a matching in $G$
- Let $C(G)$ be min cardinality of a vertex cover in $G$
- We already proved that $M(G) \leq C(G)$
- We will prove $C(G) \leq M(G)$ by induction on number of nodes in $G$.



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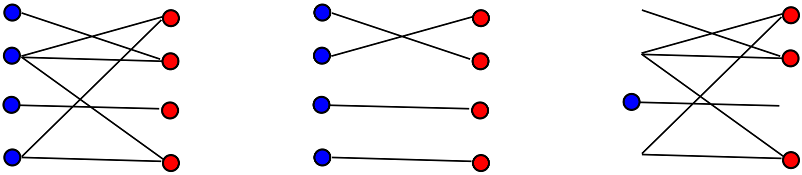

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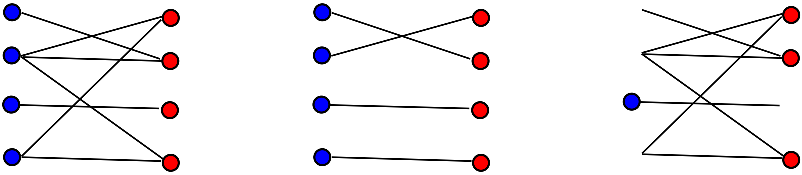

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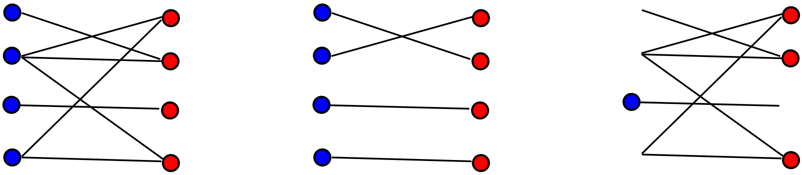

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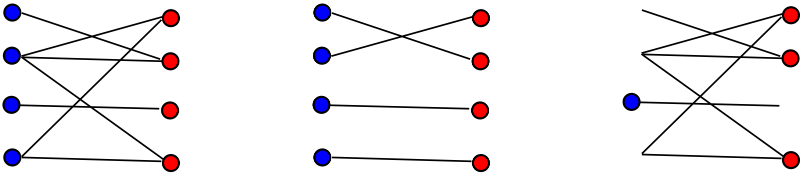

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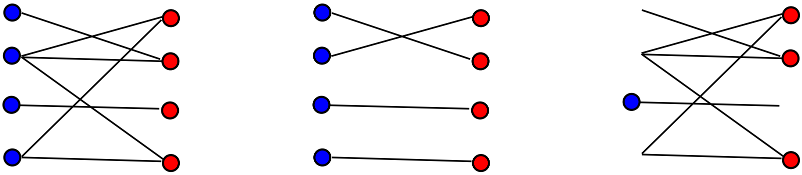

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Note: Could have proved the same using total unimodularity

# Consequences of König's Theorem

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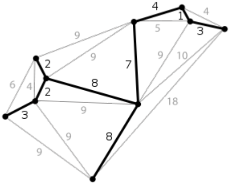
- Like maximum cardinality matching, minimum vertex cover in bipartite graphs can be formulated as an LP, and solved in polynomial time
- The same is true for the maximum independent set problem in bipartite graphs.
  - $C$ is a vertex cover iff $V \setminus C$ is an independent set.

# Outline

# The Minimum Cost Spanning Tree Problem



Given a connected undirected graph $G = (V, E)$, and costs $c_e$ on edges $e$, find a minimum cost spanning tree of $G$.

- Spanning Tree: an acyclic set of edges connecting every pair of nodes
- When graph is disconnected, can search for min-cost spanning forest instead
- We use $n$ and $m$ to denote $|V|$ and $|E|$, respectively.

# Kruskal's Algorithm

The minimum spanning tree problem can be solved efficiently by a simple greedy algorithm

## Kruskal's algorithm

1. $T = \emptyset$
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- Proof of correctness is via a simple exchange argument.
- Generalizes to Matroids

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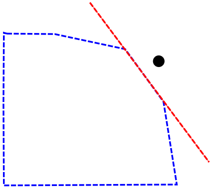
- Note: this LP has an exponential (in $n$) number of constraints

# Solving the MST Linear Program



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A separation oracle for a linear program with feasible set $\mathcal{P} \subseteq \mathbb{R}^m$ is an algorithm which takes as input $x \in \mathbb{R}^m$, and either certifies that $x \in \mathcal{P}$ or identifies a violated constraint.

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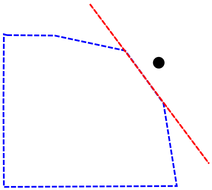

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A separation oracle for a linear program with feasible set $\mathcal{P} \subseteq \mathbb{R}^m$ is an algorithm which takes as input $x \in \mathbb{R}^m$, and either certifies that $x \in \mathcal{P}$ or identifies a violated constraint.

**Theorem**

*A linear program with a polynomial number of variables is solvable in polynomial time if and only if it admits a polynomial time separation oracle (modulo some technicalities)*

Follows from the ellipsoid method, which we will see next week.

# Solving the MST Linear Program

## Primal LP

$$\begin{array}{lll} \text{minimize} & \sum_{e \in E} c_e x_e & \\ \text{subject to} & \sum\limits_{e \subseteq X} x_e \leq |X| - 1, & \text{for } X \subset V. \\ & \sum\limits_{e \in E} x_e = n - 1 & \\ & x_e \geq 0, & \text{for } e \in E. \end{array}$$

- Given $x \in \mathbb{R}^m$, separation oracle must find a violated constraint if one exists

# Solving the MST Linear Program

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& \sum_{e \in E} x_e = n - 1 \\
& x_e \geq 0, && \text{for } e \in E.
\end{aligned}
$$

- Given $x \in \mathbb{R}^m$, separation oracle must find a violated constraint if one exists
- Reduces to finding $X \subset V$ with $\sum_{e \subseteq X} x_e > |X| - 1$, if one exists
  - Equivalently $\frac{1 + \sum_{e \subseteq X} x_e}{|X|} > 1$

# Solving the MST Linear Program

## Primal LP

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\begin{array}{lll}
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  - Equivalently $\frac{1+\sum_{e \subseteq X} x_e}{|X|} > 1$
- In turn, this reduces to maximizing $\frac{1+\sum_{e \subseteq X} x_e}{|X|}$ over $X$

We will see how to do this efficiently later in the class, using submodular minimization

# Application of Fractional Spanning Trees

- The LP formulation of spanning trees has many applications
- We will look at one contrived yet simple application that shows the flexibility enabled by polyhedral formulation

## Fault-Tolerant MST

- Your tree is an overlay network on the internet used to transmit data
- A hacker is looking to attack your tree, by knocking off one of the edges of the graph
- You can foil the hacker by choosing a random tree
- The hacker knows the algorithm you use, but not your random coins

## Fault-tolerant MST LP

$$
\begin{array}{lll}
\text{minimize} & \sum_{e \in E} c_e x_e & \\
\text{subject to} & \displaystyle\sum_{e \subseteq X} x_e \leq |X| - 1, & \text{for } X \subset V. \\
& \displaystyle\sum_{e \in E} x_e = n - 1 & \\
& x_e \leq p, & \text{for } e \in E. \\
& x_e \geq 0, & \text{for } e \in E.
\end{array}
$$

- Above LP can be solved efficiently
- If feasible, can interpret resulting fractional spanning tree $x$ as a recipe for a probability distribution over trees $T$
  - $e \in T$ with probability $x_e$
  - Since $x_e \leq p$, no edge is in the tree with probability more than $p$.

## Fault-tolerant MST LP

$$\begin{array}{lll} \text{minimize} & \sum_{e\in E} c_e x_e & \\ \text{subject to} & \sum\limits_{e\subseteq X} x_e \le |X|-1, & \text{for } X \subset V. \\ & \sum\limits_{e\in E} x_e = n-1 & \\ & x_e \le p, & \text{for } e \in E. \\ & x_e \ge 0, & \text{for } e \in E. \end{array}$$

- Given feasible solution $x$, such a probability distribution exists!

## Fault-tolerant MST LP

$$\begin{array}{lll} \text{minimize} & \sum_{e \in E} c_e x_e & \\ \text{subject to} & \sum_{e \subseteq X} x_e \leq |X| - 1, & \text{for } X \subset V. \\ & \sum_{e \in E} x_e = n - 1 & \\ & x_e \leq p, & \text{for } e \in E. \\ & x_e \geq 0, & \text{for } e \in E. \end{array}$$

- Given feasible solution $x$, such a probability distribution exists!
  - $x$ is in the (original) MST polytope
  - Caratheodory's theorem: $x$ is a convex combination of $m + 1$ vertices of MST polytope
  - By integrality of MST polytope: $x$ is the “expectation” of a probability distribution over spanning trees.

# Fault-tolerant MST LP

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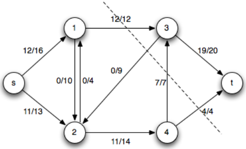
- Given feasible solution $x$, such a probability distribution exists!
  - $x$ is in the (original) MST polytope
  - Caratheodory's theorem: $x$ is a convex combination of $m + 1$ vertices of MST polytope
  - By integrality of MST polytope: $x$ is the "expectation" of a probability distribution over spanning trees.
- Consequence of Ellipsoid algorithm: can compute such a decomposition of $x$ efficiently!

# Outline




## The Maximum Flow Problem

Given a directed graph $G = (V, E)$ with capacities $u_e$ on edges $e$, a source node $s$, and a sink node $t$, find a maximum flow from $s$ to $t$ respecting the capacities.

$$
\begin{array}{lll}
\text{maximize} & \sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e & \\
\text{subject to} & \sum_{e\in\delta^-(v)} x_e = \sum_{e\in\delta^+(v)} x_e, & \text{for } v \in V \setminus \{s,t\}\,. \\
& x_e \le u_e, & \text{for } e \in E. \\
& x_e \ge 0, & \text{for } e \in E.
\end{array}
$$

Can be computed either by solving the LP, or by a combinatorial algorithm such as Ford Fulkerson.

## Primal LP

max $\sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e$

s.t.

$$\sum_{e\in\delta^-(v)} x_e = \sum_{e\in\delta^+(v)} x_e, \qquad \forall v \in V \setminus \{s,t\}$$

$$x_e \le u_e, \qquad \forall e \in E.$$

$$x_e \ge 0, \qquad \forall e \in E.$$

## Dual LP (Simplified)

min $\sum_{e\in E} u_e z_e$

s.t.

$$y_v - y_u \le z_e, \qquad \forall e = (u,v) \in E.$$

$$y_s = 0$$

$$y_t = 1$$

$$z_e \ge 0, \qquad \forall e \in E.$$

- Dual solution describes fraction $z_e$ of each edge to fractionally cut

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$$z_e \ge 0, \qquad \forall e \in E.$$

- Dual solution describes fraction $z_e$ of each edge to fractionally cut
- Dual constraints require that at least 1 edge is cut on every path from $s$ to $t$.
  - $\sum_{(u,v)\in P} z_{uv} \ge \sum_{(u,v)\in P} y_v - y_u = y_t - y_s = 1$

## Primal LP

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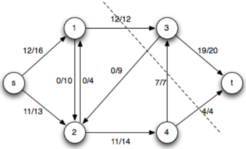


- Every integral $s - t$ cut is feasible.

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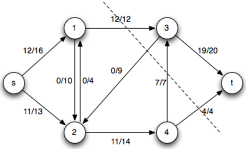


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- By weak duality: max flow $\le$ minimum cut

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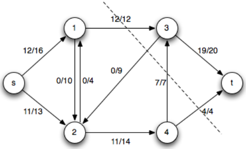


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- Ford-Fulkerson shows that max flow $=$ min cut
  - i.e. dual has integer optimal

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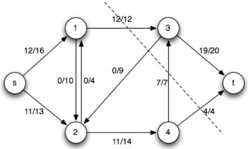


- Every integral $s-t$ cut is feasible.
- By weak duality: max flow $\le$ minimum cut
- Ford-Fulkerson shows that max flow $=$ min cut
  - i.e. dual has integer optimal
- Ford-Fulkerson also shows that there is an integral optimal flow when capacities are integer.

# Generalizations of Max Flow

$$\max \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e$$

s.t.

$$\sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \qquad \forall v \in V \setminus \{s, t\}.$$

$$x_e \leq u_e, \qquad \forall e \in E.$$

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Writing as an LP shows that many generalizations are also tractable

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- Lower and upper bound constraints on flow: $\ell_e \leq x_e \leq u_e$

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- Lower and upper bound constraints on flow: $\ell_e \leq x_e \leq u_e$
- minimum cost flow of a certain amount $r$
  - Objective $\min \sum_e c_e x_e$
  - Additional constraint: $\sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e = r$

# Generalizations of Max Flow

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- Multiple commodities sharing the network

# Generalizations of Max Flow

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- Multiple commodities sharing the network
- …

## Minimum Congestion Flow

You are given a directed graph $G = (V, E)$ with congestion functions $c_e(.)$ on edges $e$, a source node $s$, a sink node $t$, and a desired flow amount $r$. Find a minimum average congestion flow from $s$ to $t$.

$$
\begin{array}{lll}
\text{minimize} & \sum_e x_e c_e(x_e) & \\
\text{subject to} & \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e = r & \\
 & \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, & \text{for } v \in V \setminus \{s, t\}. \\
 & x_e \geq 0, & \text{for } e \in E.
\end{array}
$$

When $c_e(.)$ are polynomials with nonnegative co-efficients, e.g. $c_e(x) = a_e x^2 + b_e x + c_e$ with $a_e, b_e, c_e \geq 0$, this is a (non-linear) convex program.

# Outline

## The Max Cut Problem

Given an undirected graph $G = (V, E)$, find a partition of $V$ into $(S, V \setminus S)$ maximizing number of edges with exactly one end in $S$.

$$
\begin{array}{lll}
\text{maximize} & \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2} & \\
\text{subject to} & x_i \in \{-1, 1\}, & \text{for } i \in V.
\end{array}
$$

## The Max Cut Problem

Given an undirected graph $G = (V, E)$, find a partition of $V$ into $(S, V \setminus S)$ maximizing number of edges with exactly one end in $S$.

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Instead of requiring $x_i$ to be on the 1 dimensional sphere, we relax and permit it to be in the $n$-dimensional sphere.

## Vector Program relaxation

$$\begin{array}{lll} \text{maximize} & \sum_{(i,j)\in E} \frac{1 - \vec{v}_i \cdot \vec{v}_j}{2} & \\ \text{subject to} & ||\vec{v}_i||_2 = 1, & \text{for } i \in V. \\ & \vec{v}_i \in \mathbb{R}^n, & \text{for } i \in V. \end{array}$$

# SDP Relaxation

- Recall: A symmetric $n \times n$ matrix $Y$ is PSD iff $Y = V^T V$ for $n \times n$ matrix $V$
- Equivalently: PSD matrices encode pairwise dot products of columns of $V$
- When diagonal entries of $Y$ are $1$, $V$ has unit length columns
- Recall: $Y$ and $V$ can be recovered from each other efficiently

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- When diagonal entries of $Y$ are $1$, $V$ has unit length columns
- Recall: $Y$ and $V$ can be recovered from each other efficiently

**Vector Program relaxation**

$$
\begin{array}{lll}
\text{maximize} & \sum_{(i,j)\in E} \frac{1-\vec{v}_i\cdot\vec{v}_j}{2} & \\
\text{subject to} & ||\vec{v}_i||_2 = 1, & \text{for } i \in V. \\
 & \vec{v}_i \in \mathbb{R}^n, & \text{for } i \in V.
\end{array}
$$

**SDP Relaxation**

$$
\begin{array}{lll}
\text{maximize} & \sum_{(i,j)\in E} \frac{1-Y_{ij}}{2} & \\
\text{subject to} & Y_{ii} = 1, & \text{for } i \in V. \\
 & Y \in S^n_+ &
\end{array}
$$

## SDP Relaxation

$$\begin{array}{lll} \text{maximize} & \sum_{(i,j)\in E} \frac{1-Y_{ij}}{2} & \\ \text{subject to} & Y_{ii} = 1, & \text{for } i \in V. \\ & Y \in S_+^n & \end{array}$$

## Randomized Algorithm for Max Cut

1. Solve the SDP to get $Y \succeq 0$
2. Decompose $Y$ to $VV^T$
3. Pick a random vector $r$ on the unit sphere
4. Place all nodes $i$ with $v_i \cdot r \geq 0$ on one side of the cut, and all others on the other side

## SDP Relaxation

$$\begin{array}{lll} \text{maximize} & \sum_{(i,j)\in E} \frac{1-Y_{ij}}{2} & \\ \text{subject to} & Y_{ii} = 1, & \text{for } i \in V. \\ & Y \in S_+^n & \end{array}$$

## Randomized Algorithm for Max Cut

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## Lemma

The SDP cuts each edge with probability at least $0.878\frac{1-Y_{ij}}{2}$

Consequently, by linearity of expectation, expected number of edges cut is at least $0.878\,OPT$.

## Lemma

The SDP cuts each edge with probability at least $0.878\frac{1-Y_{ij}}{2}$

We use the following fact

## Fact

For all angles $\theta \in [0, \pi]$,

$$\frac{\theta}{\pi} \geq 0.878 \cdot \frac{1}{2}(1 - cos(\theta))$$

to prove the Lemma on the board.