

CS675: Convex and Combinatorial Optimization
Spring 2018
Convex Sets

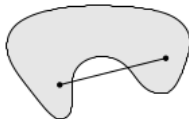
Instructor: Shaddin Dughmi

Outline

- 1 Convex sets, Affine sets, and Cones
- 2 Examples of Convex Sets
- 3 Convexity-Preserving Operations
- 4 Separation Theorems

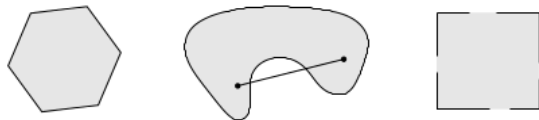
Convex Sets

A set $S \subseteq \mathbb{R}^n$ is **convex** if the line segment between any two points in S lies in S . i.e. if $x, y \in S$ and $\theta \in [0, 1]$, then $\theta x + (1 - \theta)y \in S$.



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Equivalent Definition

S is convex if every **convex combination** of points in S lies in S .

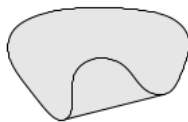
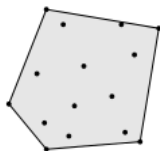
Convex Combination

- **Finite:** y is a convex combination of x_1, \dots, x_k if $y = \theta_1 x_1 + \dots + \theta_k x_k$, where $\theta_i \geq 0$ and $\sum_i \theta_i = 1$.
- **General:** expectation of probability measure on S .

Convex Hull

The convex hull of $S \subseteq \mathbb{R}^n$ is the smallest convex set containing S .

- Intersection of all convex sets containing S
- The set of all convex combinations of points in S

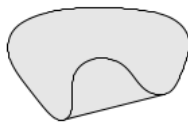
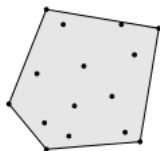


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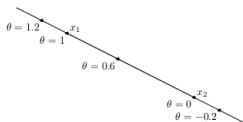
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A set S is convex if and only if $\text{convexhull}(S) = S$.

Affine Set

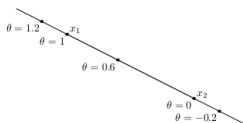
A set $S \subseteq \mathbb{R}^n$ is **affine** if the line passing through any two points in S lies in S . i.e. if $x, y \in S$ and $\theta \in \mathbb{R}$, then $\theta x + (1 - \theta)y \in S$.



Obviously, affine sets are convex.

Affine Set

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Obviously, affine sets are convex.

Equivalent Definition

S is affine if every **affine combination** of points in S lies in S .

Affine Combination

y is an affine combination of x_1, \dots, x_k if $y = \theta_1 x_1 + \dots + \theta_k x_k$, and $\sum_i \theta_i = 1$.

Generalizes convex combinations

Equivalent Definition II

S is affine if and only if it is a shifted subspace

- i.e. $S = x_0 + V$, where V is a linear subspace of \mathbb{R}^n .
- Any $x_0 \in S$ will do, and yields the same V .
- The **dimension** of S is the dimension of subspace V .

Equivalent Definition II

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Equivalent Definition III

S is affine if and only if it is the solution of a set of linear equations (i.e. the intersection of hyperplanes).

- i.e. $S = \{x : Ax = b\}$ for some matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

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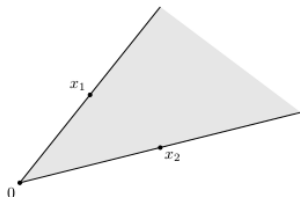
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Affine Dimension

The **affine dimension** of a set is the dimension of its affine hull

Cones

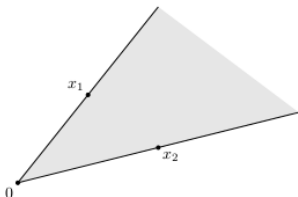
A set $K \subseteq \mathbb{R}^n$ is a **cone** if the ray from the origin through every point in K is in K i.e. if $x \in K$ and $\theta \geq 0$, then $\theta x \in K$.



Note: every cone contains 0 .

Cones

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Special Cones

- A **convex cone** is a cone that is convex
- A cone is **pointed** if whenever $x \in K$ and $x \neq 0$, then $-x \notin K$.
- We will mostly mention **proper** cones: convex, pointed, closed, and of full affine dimension.

Equivalent Definition

K is a convex cone if every **conic combination** of points in K lies in K .

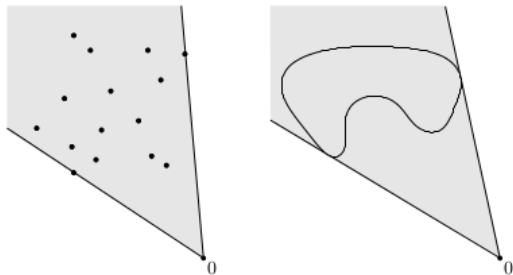
Conic Combination

y is a conic combination of x_1, \dots, x_k if $y = \theta_1 x_1 + \dots + \theta_k x_k$, where $\theta_i \geq 0$.

Conic Hull

The conic hull of $K \subseteq \mathbb{R}^n$ is the smallest convex cone containing K

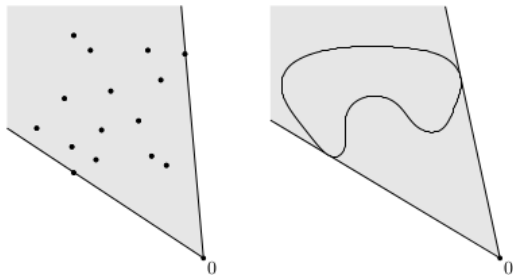
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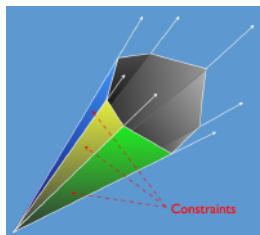
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Polyhedral Cone

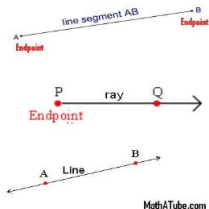
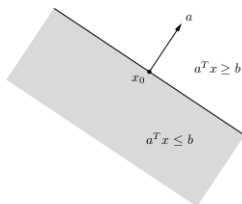
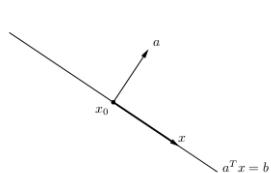
A cone is **polyhedral** if it is the set of solutions to a finite set of homogeneous linear inequalities $Ax \leq 0$.



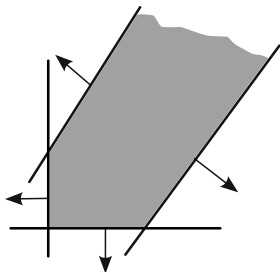
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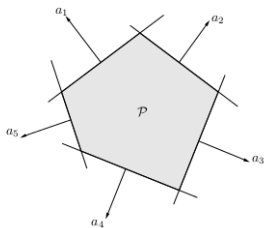
- Linear Subspace: Affine, Cone
- Hyperplane: Affine, cone if includes 0
- Halfspace: Cone if origin on boundary
- Line: Affine, cone if includes 0
- Ray: Cone if endpoint at 0
- Line segment



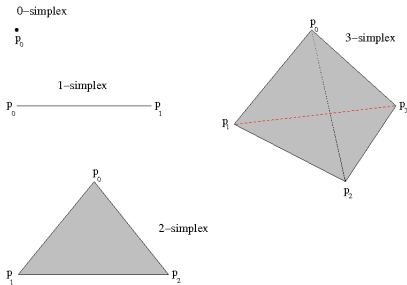
- Polyhedron: finite intersection of halfspaces



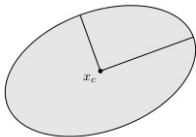
- Polytope: Bounded polyhedron



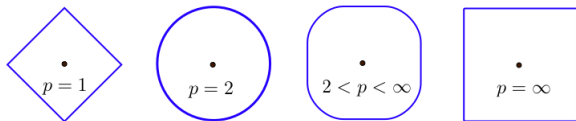
- Nonnegative Orthant \mathbb{R}_+^n : Polyhedral cone
- Simplex: convex hull of affinely independent points
 - Unit simplex: $x \succeq 0, \sum_i x_i \leq 1$
 - Probability simplex: $x \succeq 0, \sum_i x_i = 1$.



- Euclidean ball: $\{x : \|x - x_c\|_2 \leq r\}$ for center x_c and radius r
- Ellipsoid: $\{x : (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$ for symmetric $P \succeq 0$
 - Equivalently: $\{x_c + Au : \|u\|_2 \leq 1\}$ for some linear map A

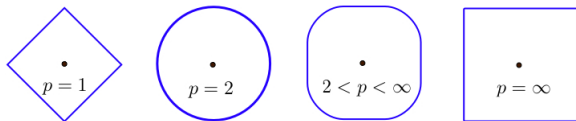


- Norm ball: $\{x : \|x - c\| \leq r\}$ for any norm $\|\cdot\|$



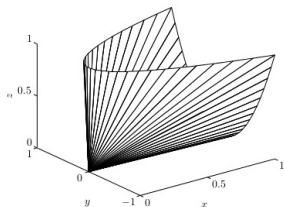
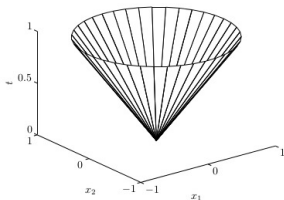
The unit sphere for different metrics: $\|x\|_{l_p} = 1$ in \mathbb{R}^2 .

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The unit sphere for different metrics: $\|x\|_{l_p} = 1$ in \mathbb{R}^2 .

- Norm cone: $\{(x, r) : \|x\| \leq r\}$
- Cone of symmetric positive semi-definite matrices M
 - Symmetric matrix $A \succeq 0$ iff $x^T A x \geq 0$ for all x

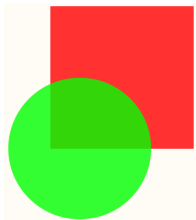


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Intersection

The intersection of two convex sets is convex. This holds for the intersection of an infinite number of sets.

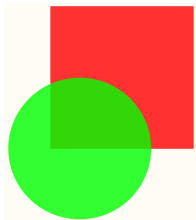


Examples

- Polyhedron: intersection of halfspaces
- PSD cone: intersection of linear inequalities $z^T A z \geq 0$, for all $z \in \mathbb{R}^n$.

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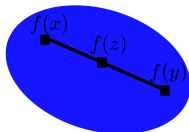
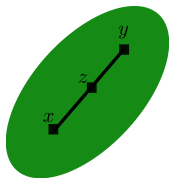
In fact, we will see that every closed convex set is the intersection of a (possibly infinite) set of halfspaces.

Affine Maps

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an affine function (i.e. $f(x) = Ax + b$), then

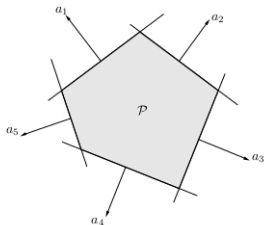
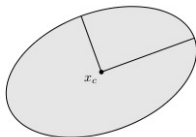
- $f(S)$ is convex whenever $S \subseteq \mathbb{R}^n$ is convex
- $f^{-1}(T)$ is convex whenever $T \subseteq \mathbb{R}^m$ is convex

$$\begin{aligned}f(\theta x + (1 - \theta)y) &= A(\theta x + (1 - \theta)y) + b \\ &= \theta(Ax + b) + (1 - \theta)(Ay + b) \\ &= \theta f(x) + (1 - \theta)f(y)\end{aligned}$$



Examples

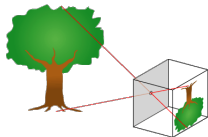
- An ellipsoid is image of a unit ball after an affine map
- A polyhedron $Ax \preceq b$ is inverse image of nonnegative orthant under $f(x) = b - Ax$



Perspective Function

Let $P : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be $P(x, t) = x/t$.

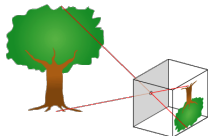
- $P(S)$ is convex whenever $S \subseteq \mathbb{R}^{n+1}$ is convex
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Generalizes to **linear fractional functions** $f(x) = \frac{Ax+b}{c^T x+d}$

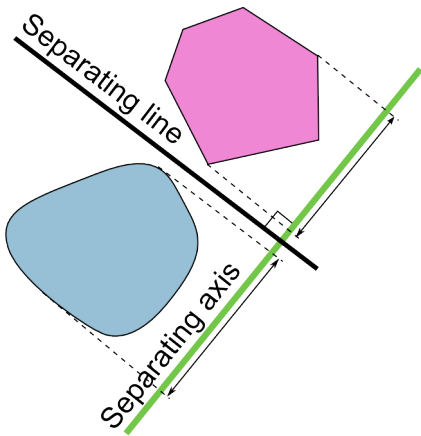
- Composition of perspective with affine.

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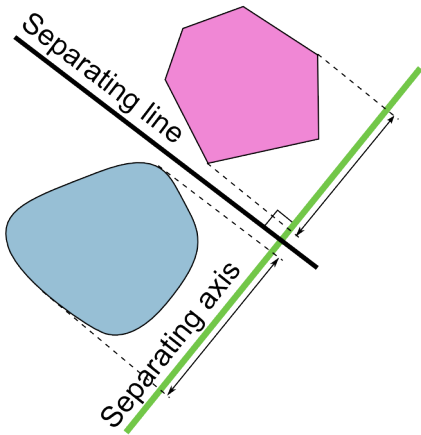
Separating Hyperplane Theorem

If $A, B \subseteq \mathbb{R}^n$ are disjoint convex sets, then there is a hyperplane **weakly** separating them. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^\top x \leq b$ for every $x \in A$ and $a^\top y \geq b$ for every $y \in B$.



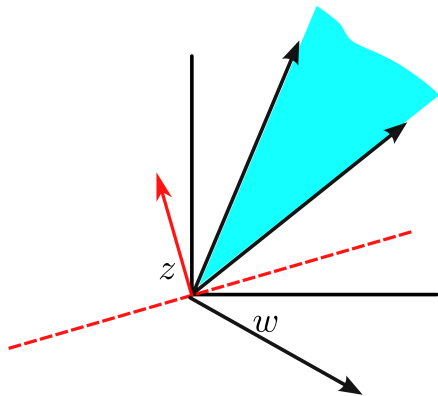
Separating Hyperplane Theorem (Strict Version)

If $A, B \subseteq \mathbb{R}^n$ are disjoint **closed** convex sets, and at least one of them is compact, then there is a hyperplane **strictly** separating them. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^\top x < b$ for every $x \in A$ and $a^\top y > b$ for every $y \in B$.



Farkas' Lemma

Let K be a **closed convex cone** and let $w \notin K$. There is $z \in \mathbb{R}^n$ such that $z^\top x \geq 0$ for all $x \in K$, and $z^\top w < 0$.



Supporting Hyperplane

Supporting Hyperplane Theorem.

If $S \subseteq \mathbb{R}^n$ is a closed convex set and y is on the boundary of S , then there is a hyperplane **supporting** S at y . That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^\top x \leq b$ for every $x \in S$ and $a^\top y = b$.

