# CS675: Convex and Combinatorial Optimization
# Spring 2018
# Consequences of the Ellipsoid Algorithm

Instructor: Shaddin Dughmi

# Outline

# Recall: Feasibility Problem

The ellipsoid method solves the following problem.

## Convex Feasibility Problem

Given as input the following

- A description of a compact convex set $K \subseteq \mathbb{R}^n$
- An ellipsoid $E(c, Q)$ (typically a ball) containing $K$
- A rational number $R > 0$ satisfying $\mathbf{vol}(E) \leq R$.
- A rational number $r > 0$ such that if $K$ is nonempty, then $\mathbf{vol}(K) \geq r$.

Find a point $x \in K$ or declare that $K$ is empty.

- Equivalent variant: drop the requirement on volume $\mathbf{vol}(K)$, and either find a point $x \in K$ or an ellipsoid $E \supseteq K$ with $\mathbf{vol}(E) < r$.

All the ellipsoid method needed was the following subroutine

### Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separting $x$ from $K$.

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\intercal x \geq h^\intercal y$ for all $y \in K$.
- Equivalently, $K$ is contained in the halfspace

$$H(h, x) = \{y : h^\intercal y \leq h^\intercal x\}$$

with $x$ at its boundary.

All the ellipsoid method needed was the following subroutine

## Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separting $x$ from $K$.

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\intercal x \geq h^\intercal y$ for all $y \in K$.
- Equivalently, $K$ is contained in the halfspace

$$H(h, x) = \{y : h^\intercal y \leq h^\intercal x\}$$

  with $x$ at its boundary.

Examples:

All the ellipsoid method needed was the following subroutine

## Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separting $x$ from $K$.

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\intercal x \geq h^\intercal y$ for all $y \in K$.
- Equivalently, $K$ is contained in the halfspace

$$H(h, x) = \{y : h^\intercal y \leq h^\intercal x\}$$

with $x$ at its boundary.

Examples:

- Explicitly written polytope $Ay \leq b$: take $h = a_i$ to the row of $A$ corresponding to a constraint violated by $x$.

All the ellipsoid method needed was the following subroutine

## Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separting $x$ from $K$.

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\intercal x \geq h^\intercal y$ for all $y \in K$.
- Equivalently, $K$ is contained in the halfspace

$$H(h, x) = \{y : h^\intercal y \leq h^\intercal x\}$$

  with $x$ at its boundary.

Examples:

- Explicitly written polytope $Ay \leq b$: take $h = a_i$ to the row of $A$ corresponding to a constraint violated by $x$.
- Convex set given by a family of convex inequalities $f_i(y) \leq 0$: Let $h = \bigtriangledown f_i(x)$ for some violated constraint.

All the ellipsoid method needed was the following subroutine

## Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separting $x$ from $K$.

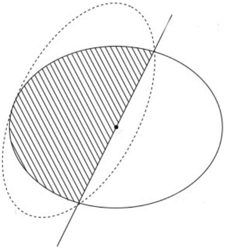
- i.e. a vector $h \in \mathbb{R}^n$ with $h^\intercal x \geq h^\intercal y$ for all $y \in K$.
- Equivalently, $K$ is contained in the halfspace

$$H(h,x) = \{y : h^\intercal y \leq h^\intercal x\}$$

  with $x$ at its boundary.

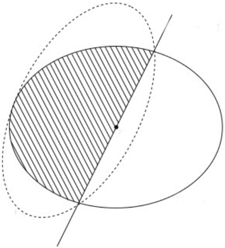
Examples:

- Explicitly written polytope $Ay \leq b$: take $h = a_i$ to the row of $A$ corresponding to a constraint violated by $x$.
- Convex set given by a family of convex inequalities $f_i(y) \leq 0$: Let $h = \bigtriangledown f_i(x)$ for some violated constraint.
- The positive semi-definite cone $S_n^+$: Let $H$ be the outer product $vv^\intercal$ of an eigenvector $v$ of $X$ corresponding to a negative eigenvalue.



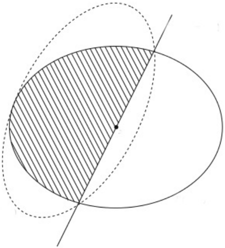
## Ellipsoid Method

1. Start with initial ellipsoid $E = E(c, Q) \supseteq K$
2. Using the separation oracle, check if the center $c \in K$.
   - If so, terminate and output $c$.
   - Otherwise, we get a separating hyperplane $h$ such that $K$ is contained in the half-ellipsoid $E \bigcap \{y : h^\intercal y \leq h^\intercal c\}$
3. Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
4. If $\mathbf{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return "empty".



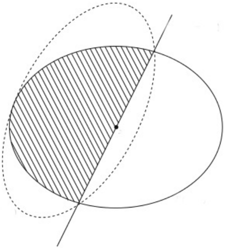
## Ellipsoid Method

1. Start with initial ellipsoid $E = E(c, Q) \supseteq K$
2. Using the separation oracle, check if the center $c \in K$.
   - If so, terminate and output $c$.
   - Otherwise, we get a separating hyperplane $h$ such that $K$ is contained in the half-ellipsoid $E \bigcap \{y : h^\intercal y \leq h^\intercal c\}$
3. Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
4. If $\mathbf{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return "empty".



## Ellipsoid Method

1. Start with initial ellipsoid $E = E(c, Q) \supseteq K$
2. Using the separation oracle, check if the center $c \in K$.
   - If so, terminate and output $c$.
   - Otherwise, we get a separating hyperplane $h$ such that $K$ is contained in the half-ellipsoid $E \bigcap \{y : h^\intercal y \leq h^\intercal c\}$
3. Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
4. If $\mathbf{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return "empty".



## Ellipsoid Method

1. Start with initial ellipsoid $E = E(c, Q) \supseteq K$
2. Using the separation oracle, check if the center $c \in K$.
   - If so, terminate and output $c$.
   - Otherwise, we get a separating hyperplane $h$ such that $K$ is contained in the half-ellipsoid $E \bigcap \{y : h^\intercal y \leq h^\intercal c\}$
3. Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
4. If $\mathbf{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return “empty”.

# Properties

Using $T$ to denote the runtime of the separation oracle

**Theorem**

The ellipsoid algorithm terminates in time polynomial $n$, $\ln \frac{R}{r}$, and $T$, and either outputes $x \in K$ or correctly declares that $K$ is empty.

We proved most of this (modulo the ellipsoid updating Lemma which we cited and briefly discussed).

# Properties

Using $T$ to denote the runtime of the separation oracle

**Theorem**

The ellipsoid algorithm terminates in time polynomial $n$, $\ln \frac{R}{r}$, and $T$, and either outputes $x \in K$ or correctly declares that $K$ is empty.

We proved most of this (modulo the ellipsoid updating Lemma which we cited and briefly discussed).

**Note**

For runtime polynomial in input size we need

- $T$ polynomial in input size
- $\frac{R}{r}$ exponential in input size

# Outline

## Recall: Convex Optimization Problem

A problem of minimizing a convex function (or maximizing a concave function) over a convex set.

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

Where $\mathcal{X} \subseteq \mathbb{R}^n$ is convex and closed, and $f : \mathbb{R}^n \to \mathbb{R}$ is convex

# Recall: Convex Optimization Problem

A problem of minimizing a convex function (or maximizing a concave function) over a convex set.

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

Where $\mathcal{X} \subseteq \mathbb{R}^n$ is convex and closed, and $f : \mathbb{R}^n \to \mathbb{R}$ is convex

- Recall: A problem $\Pi$ is a family of instances $I = (f, \mathcal{X})$
- When represented explicitly, often given in standard form

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \leq 0, & \text{for } i \in \mathcal{C}_1. \\ & a_i^\intercal x = b_i, & \text{for } i \in \mathcal{C}_2. \end{array}$$

- The functions $f, \{g_i\}_i$ are given in some parametric form allowing evaluation of each function and its derivatives.

## Recall: Convex Optimization Problem

A problem of minimizing a convex function (or maximizing a concave function) over a convex set.

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

Where $\mathcal{X} \subseteq \mathbb{R}^n$ is convex and closed, and $f : \mathbb{R}^n \to \mathbb{R}$ is convex

- We will abstract away details of how instances of a problem are represented, but denote the length of the description by $\langle I \rangle$
- Require polynomial time (in $\langle I \rangle$ and $n$) implementation of separation oracle, and other subroutines.

# Solvability of Convex Optimization

There are many subtly different "solvability statements". This one is the most useful, yet simple to describe, IMO.

## Requirements

We say an algorithm weakly solves a convex optimization problem in polynomial time if it:

- Takes an approximation parameter $\epsilon > 0$
- Terminates in time $\mathrm{poly}(\langle I \rangle, n, \log(\frac{1}{\epsilon}))$
- Returns an $\epsilon$-optimal $x \in \mathcal{X}$:

$$f(x) \leq \min_{y \in \mathcal{X}} f(y) + \epsilon[\max_{y \in \mathcal{X}} f(y) - \min_{y \in \mathcal{X}} f(y)]$$

# Solvability of Convex Optimization

## Theorem (Polynomial Solvability of CP)

*Consider a family $\Pi$ of convex optimization problems $I = (f, \mathcal{X})$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):*

- *A separation oracle for the feasible set $\mathcal{X} \subseteq \mathbb{R}^n$*
- *A first order oracle for $f$: evaluates $f(x)$ and $\bigtriangledown f(x)$.*
- *An algorithm which computes a starting ellipsoid $E \supseteq \mathcal{X}$ with $\frac{\mathbf{vol}(E)}{\mathbf{vol}(\mathcal{X})} = O(\exp(\langle I \rangle, n))$.*

*Then there is a polynomial time algorithm which weakly solves $\Pi$.*

# Solvability of Convex Optimization

## Theorem (Polynomial Solvability of CP)

*Consider a family $\Pi$ of convex optimization problems $I = (f, \mathcal{X})$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):*

- *A separation oracle for the feasible set $\mathcal{X} \subseteq \mathbb{R}^n$*
- *A first order oracle for $f$: evaluates $f(x)$ and $\bigtriangledown f(x)$.*
- *An algorithm which computes a starting ellipsoid $E \supseteq \mathcal{X}$ with $\frac{\mathbf{vol}(E)}{\mathbf{vol}(\mathcal{X})} = O(\exp(\langle I \rangle, n))$.*

*Then there is a polynomial time algorithm which weakly solves $\Pi$.*

Let's now prove this, by reducing to the ellipsoid method

# Proof (Simplified)

## Simplifying Assumption

Assume we are given $\min_{y \in \mathcal{X}} f(y)$ and $\max_{y \in \mathcal{X}} f(y)$. Without loss of generality assume they are $[0, 1]$.

# Proof (Simplified)

## Simplifying Assumption

Assume we are given $\min_{y \in \mathcal{X}} f(y)$ and $\max_{y \in \mathcal{X}} f(y)$. Without loss of generality assume they are $[0, 1]$.

Our task reduces to the following convex feasibility problem:

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & x \in \mathcal{X} \\ & f(x) \leq \epsilon \end{array}$$

# Proof (Simplified)

## Simplifying Assumption

Assume we are given $\min_{y \in \mathcal{X}} f(y)$ and $\max_{y \in \mathcal{X}} f(y)$. Without loss of generality assume they are $[0, 1]$.

Our task reduces to the following convex feasibility problem:

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & x \in \mathcal{X} \\ & f(x) \leq \epsilon \end{array}$$

We can feed this into the Ellipsoid method!

## Needed Ingredients

1. Separation oracle for new feasible set $K$:
2. Ellipsoid $E$ containing $K$:
3. Guarantee that $\frac{\mathbf{vol}(E)}{\mathbf{vol}(K)} \leq \exp(n, \langle I \rangle, \log \frac{1}{\epsilon})$:

# Proof (Simplified)

## Simplifying Assumption

Assume we are given $\min_{y \in \mathcal{X}} f(y)$ and $\max_{y \in \mathcal{X}} f(y)$. Without loss of generality assume they are $[0, 1]$.

Our task reduces to the following convex feasibility problem:

$$\begin{aligned} \text{find} \quad & x \\ \text{subject to} \quad & x \in \mathcal{X} \\ & f(x) \leq \epsilon \end{aligned}$$

We can feed this into the Ellipsoid method!

## Needed Ingredients

1. Separation oracle for new feasible set $K$: Use the separation oracle for $\mathcal{X}$ and first order oracle for $f$
2. Ellipsoid $E$ containing $K$:
3. Guarantee that $\frac{\mathbf{vol}(E)}{\mathbf{vol}(K)} \leq \exp(n, \langle I \rangle, \log \frac{1}{\epsilon})$:

# Proof (Simplified)

## Simplifying Assumption

Assume we are given $\min_{y \in \mathcal{X}} f(y)$ and $\max_{y \in \mathcal{X}} f(y)$. Without loss of generality assume they are $[0, 1]$.

Our task reduces to the following convex feasibility problem:

$$\begin{aligned} &\text{find} && x \\ &\text{subject to} && x \in \mathcal{X} \\ & && f(x) \leq \epsilon \end{aligned}$$

We can feed this into the Ellipsoid method!

## Needed Ingredients

1. Separation oracle for new feasible set $K$: Use the separation oracle for $\mathcal{X}$ and first order oracle for $f$
2. Ellipsoid $E$ containing $K$: Use the ellipsoid containing $\mathcal{X}$
3. Guarantee that $\frac{\mathbf{vol}(E)}{\mathbf{vol}(K)} \leq \exp(n, \langle I \rangle, \log \frac{1}{\epsilon})$:

# Proof (Simplified)

## Simplifying Assumption

Assume we are given $\min_{y \in \mathcal{X}} f(y)$ and $\max_{y \in \mathcal{X}} f(y)$. Without loss of generality assume they are $[0, 1]$.

Our task reduces to the following convex feasibility problem:

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & x \in \mathcal{X} \\ & f(x) \leq \epsilon \end{array}$$

We can feed this into the Ellipsoid method!

## Needed Ingredients

1. Separation oracle for new feasible set $K$: Use the separation oracle for $\mathcal{X}$ and first order oracle for $f$
2. Ellipsoid $E$ containing $K$: Use the ellipsoid containing $\mathcal{X}$
3. Guarantee that $\frac{\mathbf{vol}(E)}{\mathbf{vol}(K)} \leq \exp(n, \langle I \rangle, \log \frac{1}{\epsilon})$: Not obvious, but true!

# Proof (Simplified)

$$K = \{x \in \mathcal{X} : f(x) \leq \epsilon\}$$

**Lemma**

$\mathbf{vol}(K) \geq \epsilon^n \, \mathbf{vol}(X)$.

This shows that $\mathbf{vol}(K)$ is only exponentially smaller (in $n$ and $\log \frac{1}{\epsilon}$) than $\mathbf{vol}(\mathcal{X})$, and therefore also $\mathbf{vol}(E)$, so it suffices.

# Proof (Simplified)

$$K = \{x \in \mathcal{X} : f(x) \leq \epsilon\}$$

**Lemma**

$\mathbf{vol}(K) \geq \epsilon^n \, \mathbf{vol}(X)$.

This shows that $\mathbf{vol}(K)$ is only exponentially smaller (in $n$ and $\log \frac{1}{\epsilon}$) than $\mathbf{vol}(\mathcal{X})$, and therefore also $\mathbf{vol}(E)$, so it suffices.

- Assume wlog $0 \in \mathcal{X}$ and $f(0) = \min_{x \in \mathcal{X}} f(x) = 0$.

# Proof (Simplified)

$$K = \{x \in \mathcal{X} : f(x) \leq \epsilon\}$$

**Lemma**

$\mathbf{vol}(K) \geq \epsilon^n \, \mathbf{vol}(X)$.

This shows that $\mathbf{vol}(K)$ is only exponentially smaller (in $n$ and $\log \frac{1}{\epsilon}$) than $\mathbf{vol}(\mathcal{X})$, and therefore also $\mathbf{vol}(E)$, so it suffices.

- Assume wlog $0 \in \mathcal{X}$ and $f(0) = \min_{x \in \mathcal{X}} f(x) = 0$.
- Consider scaling $\mathcal{X}$ by $\epsilon$ to get $\epsilon\mathcal{X}$.
- $\mathbf{vol}(\epsilon\mathcal{X}) = \epsilon^n \, \mathbf{vol}(X)$.

# Proof (Simplified)

$$K = \{x \in \mathcal{X} : f(x) \leq \epsilon\}$$

**Lemma**

$\mathbf{vol}(K) \geq \epsilon^n \, \mathbf{vol}(X)$.

This shows that $\mathbf{vol}(K)$ is only exponentially smaller (in $n$ and $\log \frac{1}{\epsilon}$) than $\mathbf{vol}(\mathcal{X})$, and therefore also $\mathbf{vol}(E)$, so it suffices.

- Assume wlog $0 \in \mathcal{X}$ and $f(0) = \min_{x \in \mathcal{X}} f(x) = 0$.
- Consider scaling $\mathcal{X}$ by $\epsilon$ to get $\epsilon\mathcal{X}$.
- $\mathbf{vol}(\epsilon\mathcal{X}) = \epsilon^n \, \mathbf{vol}(X)$.
- We show that $\epsilon\mathcal{X} \subseteq K$ by showing $f(y) \leq \epsilon$ for all $y \in \epsilon\mathcal{X}$.

# Proof (Simplified)

$$K = \{x \in \mathcal{X} : f(x) \leq \epsilon\}$$

**Lemma**

$\mathbf{vol}(K) \geq \epsilon^n \, \mathbf{vol}(X)$.

This shows that $\mathbf{vol}(K)$ is only exponentially smaller (in $n$ and $\log \frac{1}{\epsilon}$) than $\mathbf{vol}(\mathcal{X})$, and therefore also $\mathbf{vol}(E)$, so it suffices.

- Assume wlog $0 \in \mathcal{X}$ and $f(0) = \min_{x \in \mathcal{X}} f(x) = 0$.
- Consider scaling $\mathcal{X}$ by $\epsilon$ to get $\epsilon\mathcal{X}$.
- $\mathbf{vol}(\epsilon\mathcal{X}) = \epsilon^n \, \mathbf{vol}(X)$.
- We show that $\epsilon\mathcal{X} \subseteq K$ by showing $f(y) \leq \epsilon$ for all $y \in \epsilon\mathcal{X}$.
- Let $y = \epsilon x$ for $x \in \mathcal{X}$, and invoke Jensen's inequality

$$f(y) = f(\epsilon x + (1 - \epsilon)0) \leq \epsilon f(x) + (1 - \epsilon) f(0) \leq \epsilon$$

# Proof (General)

- Denote $L = \min_{y \in \mathcal{X}} f(y)$ and $H = \max_{y \in \mathcal{X}} f(y)$
- If we knew the target $T = L + \epsilon[H - L]$, we can reduce to solving the feasibility problem over $K = \{x \in \mathcal{X} : f(x) \leq T\}$.

# Proof (General)

- Denote $L = \min_{y \in \mathcal{X}} f(y)$ and $H = \max_{y \in \mathcal{X}} f(y)$
- If we knew the target $T = L + \epsilon[H - L]$, we can reduce to solving the feasibility problem over $K = \{x \in \mathcal{X} : f(x) \leq T\}$.
- If we knew it lied in a sufficiently narrow range, we could binary search for $T$

# Proof (General)

- Denote $L = \min_{y \in \mathcal{X}} f(y)$ and $H = \max_{y \in \mathcal{X}} f(y)$
- If we knew the target $T = L + \epsilon[H - L]$, we can reduce to solving the feasibility problem over $K = \{x \in \mathcal{X} : f(x) \leq T\}$.
- If we knew it lied in a sufficiently narrow range, we could binary search for $T$
- We don't need to know anything about $T$!

**Key Observation**

We don't really need to know $T$, $H$, or $L$ to simulate the same execution of the ellipsoid method on $K$!!

# Proof (General)

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & x \in \mathcal{X} \\ & f(x) \leq T = L + \epsilon[H - L] \end{array}$$

- Simulate the execution of the ellipsoid method on $K$
- Polynomial number of iterations, terminating with point in $K$

# Proof (General)

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & x \in \mathcal{X} \\ & f(x) \leq T = L + \epsilon[H - L] \end{array}$$

- Simulate the execution of the ellipsoid method on $K$
- Polynomial number of iterations, terminating with point in $K$
- Require separation oracle for $K$ to use $\bigtriangledown f$ only as a last resort
  - This is allowed.
  - Tries to get feasibility whenever possible.

# Proof (General)

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & x \in \mathcal{X} \\ & f(x) \leq T = L + \epsilon[H - L] \end{array}$$

- Simulate the execution of the ellipsoid method on $K$
- Polynomial number of iterations, terminating with point in $K$
- Require separation oracle for $K$ to use $\bigtriangledown f$ only as a last resort
  - This is allowed.
  - Tries to get feasibility whenever possible.
- Action of algorithm in each iteration other than the last can be described independently of $T$
  - If ellipsoid center $c \notin \mathcal{X}$, use separating hyperplane with $\mathcal{X}$.
  - Else use $\bigtriangledown f(c)$

# Proof (General)

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & x \in \mathcal{X} \\ & f(x) \leq T = L + \epsilon[H - L] \end{array}$$

- Simulate the execution of the ellipsoid method on $K$
- Polynomial number of iterations, terminating with point in $K$
- Require separation oracle for $K$ to use $\bigtriangledown f$ only as a last resort
  - This is allowed.
  - Tries to get feasibility whenever possible.
- Action of algorithm in each iteration other than the last can be described independently of $T$
  - If ellipsoid center $c \notin \mathcal{X}$, use separating hyperplane with $\mathcal{X}$.
  - Else use $\bigtriangledown f(c)$
- Run this simulation until enough iterations have passed, and take the best feasible point encountered. This must be in $K$.

# Outline

# Recall: Linear Programming

## Recall: Linear Programming Problem

A problem of maximizing a linear function over a polyhedron.

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \preceq b \end{array}$$

# Recall: Linear Programming

## Recall: Linear Programming Problem

A problem of maximizing a linear function over a polyhedron.

$$\begin{array}{ll} \text{maximize} & c^{\intercal} x \\ \text{subject to} & Ax \preceq b \end{array}$$

- When stated in standard form, optimal solution occurs at a vertex.

# Recall: Linear Programming

## Recall: Linear Programming Problem

A problem of maximizing a linear function over a polyhedron.

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \preceq b \end{array}$$

- When stated in standard form, optimal solution occurs at a vertex.
- We will consider both explicitly and implicit LPs
  - Explicit: given by $A$, $b$ and $c$
  - Implicit: Given by $c$ and a separation oracle for $Ax \leq b$.

# Recall: Linear Programming

## Recall: Linear Programming Problem

A problem of maximizing a linear function over a polyhedron.

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \preceq b \end{array}$$

- When stated in standard form, optimal solution occurs at a vertex.
- We will consider both explicitly and implicit LPs
  - Explicit: given by $A$, $b$ and $c$
  - Implicit: Given by $c$ and a separation oracle for $Ax \leq b$.
- In both cases, we require all numbers to be rational

# Recall: Linear Programming

## Recall: Linear Programming Problem

A problem of maximizing a linear function over a polyhedron.

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \preceq b \end{array}$$

- When stated in standard form, optimal solution occurs at a vertex.
- We will consider both explicitly and implicit LPs
  - Explicit: given by $A$, $b$ and $c$
  - Implicit: Given by $c$ and a separation oracle for $Ax \leq b$.
- In both cases, we require all numbers to be rational
- In the explicit case, we require polynomial time in $\langle A \rangle$, $\langle b \rangle$, and $\langle c \rangle$, the number of bits used to represent the parameters of the LP.

# Recall: Linear Programming

## Recall: Linear Programming Problem

A problem of maximizing a linear function over a polyhedron.

$$\begin{array}{ll} \text{maximize} & c^{\intercal} x \\ \text{subject to} & Ax \preceq b \end{array}$$

- When stated in standard form, optimal solution occurs at a vertex.
- We will consider both explicitly and implicit LPs
  - Explicit: given by $A$, $b$ and $c$
  - Implicit: Given by $c$ and a separation oracle for $Ax \leq b$.
- In both cases, we require all numbers to be rational
- In the explicit case, we require polynomial time in $\langle A \rangle$, $\langle b \rangle$, and $\langle c \rangle$, the number of bits used to represent the parameters of the LP.
- In the implicit case, we require polynomial time in the bit complexity of individual entries of $A, b, c$.

### Theorem (Polynomial Solvability of Explicit LP)

*There is a polynomial time algorithm for linear programming, when the linear program is represented explicitly.*

### Proof Sketch (Informal)

Using result for weakly solving convex programs, we need 4 things:

- A separation oracle for $Ax \leq b$: trivial when explicitly represented
- A first order oracle for $c^\intercal x$: also trivial
- A bounding ellipsoid of volume at most an exponential times the volume of the feasible polyhedron: tricky
- A way of "rounding" an $\epsilon$-optimal solution to an optimal vertex solution: tricky

## Theorem (Polynomial Solvability of Explicit LP)

*There is a polynomial time algorithm for linear programming, when the linear program is represented explicitly.*

## Proof Sketch (Informal)

Using result for weakly solving convex programs, we need 4 things:

- A separation oracle for $Ax \leq b$: trivial when explicitly represented
- A first order oracle for $c^\intercal x$: also trivial
- A bounding ellipsoid of volume at most an exponential times the volume of the feasible polyhedron: tricky
- A way of "rounding" an $\epsilon$-optimal solution to an optimal vertex solution: tricky

Solution to both issues involves tedious accounting of numerical issues

# Ellipsoid and Volume Bound (Informal)

Key to tackling both difficulties is the following observation:

**Lemma**

*Let $v$ be vertex of the polyhedron $Ax \leq b$. It is the case that $v$ has polynomial bit complexity, i.e. $\langle v \rangle \leq M$, where $M = O(\text{poly}(\langle A \rangle, \langle b \rangle))$.*

Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

# Ellipsoid and Volume Bound (Informal)

Key to tackling both difficulties is the following observation:

**Lemma**

*Let $v$ be vertex of the polyhedron $Ax \leq b$. It is the case that $v$ has polynomial bit complexity, i.e. $\langle v \rangle \leq M$, where $M = O(\text{poly}(\langle A \rangle, \langle b \rangle))$.*

Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

- Bounding ellipsoid: all vertices contained in the box $-2^M \leq x \leq 2^M$, which in turn is contained in an ellipsoid of volume exponential in $M$ and $n$.

# Ellipsoid and Volume Bound (Informal)

Key to tackling both difficulties is the following observation:

**Lemma**

*Let $v$ be vertex of the polyhedron $Ax \leq b$. It is the case that $v$ has polynomial bit complexity, i.e. $\langle v \rangle \leq M$, where $M = O(\text{poly}(\langle A \rangle, \langle b \rangle))$.*

Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

- To guarantee volume lowerbound, need to instead solve a "relaxed problem". Specifically, relaxing to $Ax \leq b + \epsilon$, for sufficiently small $\epsilon$ with $\langle \epsilon \rangle = \text{poly}(M)$. Gives volume exponentially small in $M$, but no smaller. Still close enough to original polyhedron so solution to relaxed problem can be "rounded" to solution of the original problem.

# Ellipsoid and Volume Bound (Informal)

Key to tackling both difficulties is the following observation:

**Lemma**

*Let $v$ be vertex of the polyhedron $Ax \leq b$. It is the case that $v$ has polynomial bit complexity, i.e. $\langle v \rangle \leq M$, where $M = O(\text{poly}(\langle A \rangle, \langle b \rangle))$.*

Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

- Rounding to a vertex: If a point $y$ is $\epsilon$-optimal for the $\epsilon$-relaxed problem, for sufficiently small $\epsilon$ chosen carefully to polynomial in description of input, then rounding to the nearest $x$ with $M$ bits recovers the vertex.

## Theorem (Polynomial Solvability of Implicit LP)

*Consider a family $\Pi$ of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):*

- *A separation oracle for the polyhedron $Ax \leq b$*
- *Explicit access to $c$*

*Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\mathrm{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for $\Pi$ (both primal and dual*).*

## Informal Proof Sketch (Primal)

## Theorem (Polynomial Solvability of Implicit LP)

*Consider a family $\Pi$ of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):*

- *A separation oracle for the polyhedron $Ax \leq b$*
- *Explicit access to $c$*

*Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\mathrm{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for $\Pi$ (both primal and dual*).*

## Informal Proof Sketch (Primal)

- Separation oracle and first order oracle are given

## Theorem (Polynomial Solvability of Implicit LP)

*Consider a family $\Pi$ of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):*

- *A separation oracle for the polyhedron $Ax \leq b$*
- *Explicit access to $c$*

*Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\mathrm{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for $\Pi$ (both primal and dual*).*

## Informal Proof Sketch (Primal)

- Separation oracle and first order oracle are given
- Rounding to a vertex exactly as in the explicit case.
  - Every vertex $v$ still has polynomial bit complexity $M$

## Theorem (Polynomial Solvability of Implicit LP)

*Consider a family $\Pi$ of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):*

- *A separation oracle for the polyhedron $Ax \leq b$*
- *Explicit access to $c$*

*Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\mathrm{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for $\Pi$ (both primal and dual*).*

## Informal Proof Sketch (Primal)

- Separation oracle and first order oracle are given
- Rounding to a vertex exactly as in the explicit case.
  - Every vertex $v$ still has polynomial bit complexity $M$
- Bounding ellipsoid: Still true that we get a bounding ellipsoid of volume exponential in $\langle I \rangle$ and $n$

## Theorem (Polynomial Solvability of Implicit LP)

*Consider a family $\Pi$ of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):*

- *A separation oracle for the polyhedron $Ax \leq b$*
- *Explicit access to $c$*

*Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\mathrm{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for $\Pi$ (both primal and dual*).*

## Informal Proof Sketch (Primal)

- Separation oracle and first order oracle are given
- Rounding to a vertex exactly as in the explicit case.
  - Every vertex $v$ still has polynomial bit complexity $M$
- Bounding ellipsoid: Still true that we get a bounding ellipsoid of volume exponential in $\langle I \rangle$ and $n$
- However, no lowerbound on the volume of $Ax \leq b$, and can't relax to $Ax \leq b + \epsilon$ as in the explicit case. (In fact, volume may be zero!)

## Theorem (Polynomial Solvability of Implicit LP)

*Consider a family $\Pi$ of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):*

- *A separation oracle for the polyhedron $Ax \leq b$*
- *Explicit access to $c$*

*Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\mathrm{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for $\Pi$ (both primal and dual*).*

## Informal Proof Sketch (Primal)

- Separation oracle and first order oracle are given
- Rounding to a vertex exactly as in the explicit case.
  - Every vertex $v$ still has polynomial bit complexity $M$
- Bounding ellipsoid: Still true that we get a bounding ellipsoid of volume exponential in $\langle I \rangle$ and $n$
- However, no lowerbound on the volume of $Ax \leq b$, and can't relax to $Ax \leq b + \epsilon$ as in the explicit case. (In fact, volume may be zero!)
  - It turns out this is still OK, but takes a lot of work.

## Theorem (Polynomial Solvability of Implicit LP)

*Consider a family $\Pi$ of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and $n$):*

- *A separation oracle for the polyhedron $Ax \leq b$*
- *Explicit access to $c$*

*Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\mathrm{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for $\Pi$ (both primal and dual*).*

For the dual, we need equivalence of separation and optimization. Also, we necessarily get a solution to a normalized version of the dual. (HW)

# Outline

# Separation and Optimization

- One interpretation of the previous theorem is that optimization of linear functions over a polytope of polynomial bit complexity reduces to implementing a separation oracle
- As it turns out, the two tasks are polynomial-time equivalent.

# Separation and Optimization

- One interpretation of the previous theorem is that optimization of linear functions over a polytope of polynomial bit complexity reduces to implementing a separation oracle
- As it turns out, the two tasks are polynomial-time equivalent.

Lets formalize the two questions, parametrized by a polytope $P$.

## Linear Optimization Problem

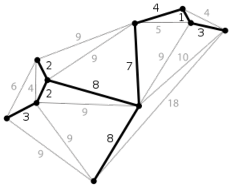
- Input: Linear objective $c \in \mathbb{R}^n$.
- Output: $\text{argmax}_{x \in P} c^\intercal x$.

## Separation Problem

- Input: $y \in \mathbb{R}^n$
- Output: Decide that $y \in P$, or else find $h \in \mathbb{R}^n$ s.t. $h^\intercal x < h^\intercal y$ for all $x \in P$.

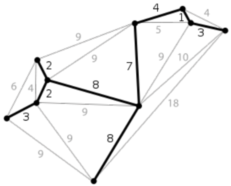
# Recall: Minimum Cost Spanning Tree

Given a connected undirected graph $G = (V, E)$, and costs $c_e$ on edges $e$, find a minimum cost spanning tree of $G$.



# Recall: Minimum Cost Spanning Tree

Given a connected undirected graph $G = (V, E)$, and costs $c_e$ on edges $e$, find a minimum cost spanning tree of $G$.

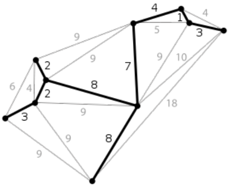



## Spanning Tree Polytope

$$\sum_{e \subseteq X} x_e \leq |X| - 1, \quad \text{for } X \subset V.$$
$$\sum_{e \in E} x_e = n - 1$$
$$x_e \geq 0, \quad \text{for } e \in E.$$

# Recall: Minimum Cost Spanning Tree

Given a connected undirected graph $G = (V, E)$, and costs $c_e$ on edges $e$, find a minimum cost spanning tree of $G$.




## Spanning Tree Polytope

$$
\begin{aligned}
&\sum_{e \subseteq X} x_e \leq |X| - 1, && \text{for } X \subset V. \\
&\sum_{e \in E} x_e = n - 1 && \\
&x_e \geq 0, && \text{for } e \in E.
\end{aligned}
$$

- Optimization: Find the minimum/maximum weight spanning tree
- Separation: Find $X \subset V$ with $\sum_{e \subseteq X} x_e > |X| - 1$, if one exists
  - i.e. When edge weights are $x$, find a “dense” subgraph

## Theorem (Equivalence of Separation and Optimization for Polytopes)

*Consider a family $\mathcal{P}$ of polytopes $P = \{x : Ax \leq b\}$ described implicitly using $\langle P \rangle$ bits, and satisfying $\langle a_{ij} \rangle, \langle b_i \rangle \leq \mathrm{poly}(\langle P \rangle, n)$. Then the separation problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle y \rangle)$ time for $P \in \mathcal{P}$ if and only if the linear optimization problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle c \rangle)$ time.*

- Colloquially, we say such polytope families are solvable.

## Theorem (Equivalence of Separation and Optimization for Polytopes)

*Consider a family $\mathcal{P}$ of polytopes $P = \{x : Ax \leq b\}$ described implicitly using $\langle P \rangle$ bits, and satisfying $\langle a_{ij} \rangle, \langle b_i \rangle \leq \mathrm{poly}(\langle P \rangle, n)$. Then the separation problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle y \rangle)$ time for $P \in \mathcal{P}$ if and only if the linear optimization problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle c \rangle)$ time.*

- Colloquially, we say such polytope families are solvable.
- E.g. Spanning tree polytopes, represented by graphs, are solvable.

## Theorem (Equivalence of Separation and Optimization for Polytopes)

*Consider a family $\mathcal{P}$ of polytopes $P = \{x : Ax \leq b\}$ described implicitly using $\langle P \rangle$ bits, and satisfying $\langle a_{ij} \rangle, \langle b_i \rangle \leq \mathrm{poly}(\langle P \rangle, n)$. Then the separation problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle y \rangle)$ time for $P \in \mathcal{P}$ if and only if the linear optimization problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle c \rangle)$ time.*

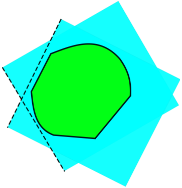
- Colloquially, we say such polytope families are solvable.
- E.g. Spanning tree polytopes, represented by graphs, are solvable.
- We already sketched the proof of the forward direction
  - Separation $\Rightarrow$ optimization

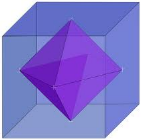
## Theorem (Equivalence of Separation and Optimization for Polytopes)

*Consider a family $\mathcal{P}$ of polytopes $P = \{x : Ax \leq b\}$ described implicitly using $\langle P \rangle$ bits, and satisfying $\langle a_{ij} \rangle, \langle b_i \rangle \leq \mathrm{poly}(\langle P \rangle, n)$. Then the separation problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle y \rangle)$ time for $P \in \mathcal{P}$ if and only if the linear optimization problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle c \rangle)$ time.*

- Colloquially, we say such polytope families are solvable.
- E.g. Spanning tree polytopes, represented by graphs, are solvable.
- We already sketched the proof of the forward direction
  - Separation $\Rightarrow$ optimization
- For the other direction, we need polars

# Recall: Polar Duality of Convex Sets





One way of representing the all halfspaces containing a convex set.

## Polar

Let $S \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The polar of $S$ is defined as follows:

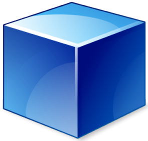
$$S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$$

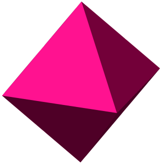
## Note

- Every halfspace $a^\intercal x \leq b$ with $b \neq 0$ can be written as a "normalized" inequality $y^\intercal x \leq 1$, by dividing by $b$.
- $S^\circ$ can be thought of as the normalized representations of halfspaces containing $S$.

## Properties of the Polar

1. If $S$ is bounded and $0 \in interior(S)$, then the same holds for $S^\circ$.
2. $S^{\circ\circ} = S$

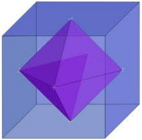




$S = \{x : y \cdot x \leq 1$ for all $y \in S^\circ\}$

$S^\circ = \{y : x \cdot y \leq 1$ for all $x \in S\}$

# Polarity of Polytopes

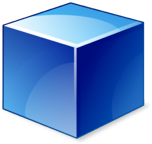


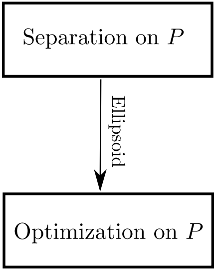
## Polytopes

Given a polytope $P$ represented as $Ax \preceq \vec{1}$, the polar $P^\circ$ is the convex hull of the rows of $A$.

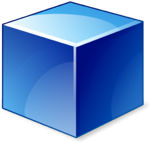
- Facets of $P$ correspond to vertices of $P^\circ$.
- Dually, vertices of $P$ correspond to facets of $P^\circ$.

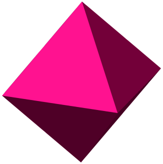
# Proof Outline: Optimization ⇒ Separation

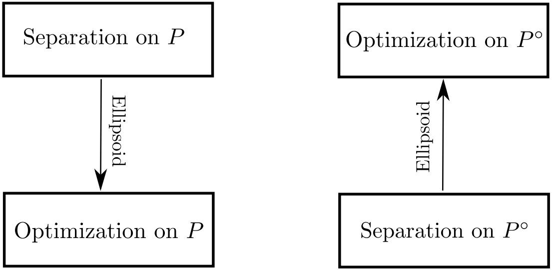




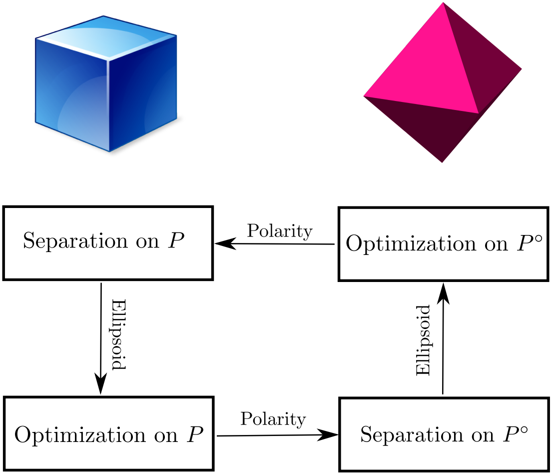
# Proof Outline: Optimization $\Rightarrow$ Separation

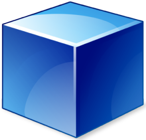


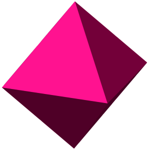




# Proof Outline: Optimization ⇒ Separation

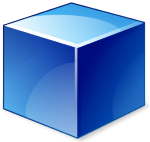


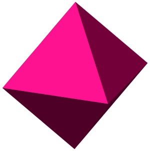




$S = \{x : y \cdot x \leq 1 \text{ for all } y \in S^\circ\}$

$S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$



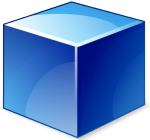


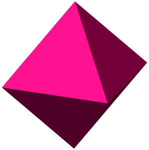
$S = \{x : y \cdot x \leq 1 \text{ for all } y \in S^\circ\}$

$S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$

**Lemma**

Separation over $S$ reduces in constant time to optimization over $S^\circ$, and vice versa since $S^{\circ\circ} = S$.





$S = \{x : y \cdot x \leq 1$ for all $y \in S^\circ\}$

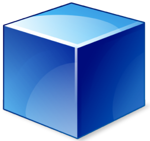
$S^\circ = \{y : x \cdot y \leq 1$ for all $x \in S\}$

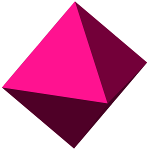
### Lemma

Separation over $S$ reduces in constant time to optimization over $S^\circ$, and vice versa since $S^{\circ\circ} = S$.

### Proof

- We are given vector $x$, and must check whether $x \in S$, and if not output separating hyperplane.





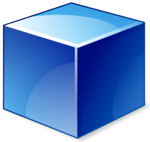
$S = \{x : y \cdot x \leq 1 \text{ for all } y \in S^\circ\}$ $\quad$ $S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$

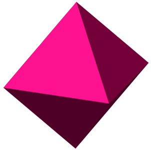
**Lemma**

Separation over $S$ reduces in constant time to optimization over $S^\circ$, and vice versa since $S^{\circ\circ} = S$.

**Proof**

- We are given vector $x$, and must check whether $x \in S$, and if not output separating hyperplane.
- $x \in S$ iff $y \cdot x \leq 1$ for all $y \in S^\circ$





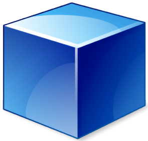
$S = \{x : y \cdot x \leq 1 \text{ for all } y \in S^\circ\}$ $S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$

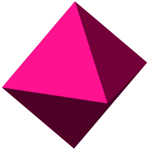
## Lemma

Separation over $S$ reduces in constant time to optimization over $S^\circ$, and vice versa since $S^{\circ\circ} = S$.

## Proof

- We are given vector $x$, and must check whether $x \in S$, and if not output separating hyperplane.
- $x \in S$ iff $y \cdot x \leq 1$ for all $y \in S^\circ$
- equivalently, iff $\max_{y \in S^\circ} y \cdot x \leq 1$.





$S = \{x : y \cdot x \leq 1$ for all $y \in S^\circ\}$ $\quad$ $S^\circ = \{y : x \cdot y \leq 1$ for all $x \in S\}$

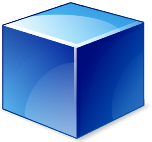
### Lemma

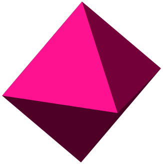
Separation over $S$ reduces in constant time to optimization over $S^\circ$, and vice versa since $S^{\circ\circ} = S$.

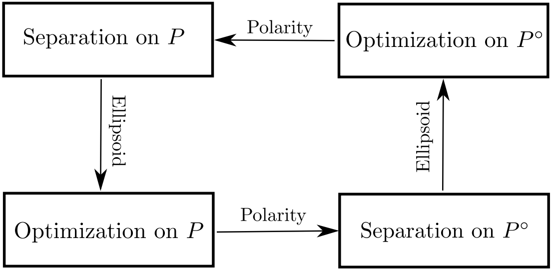
### Proof

- We are given vector $x$, and must check whether $x \in S$, and if not output separating hyperplane.
- $x \in S$ iff $y \cdot x \leq 1$ for all $y \in S^\circ$
- equivalently, iff $\max_{y \in S^\circ} y \cdot x \leq 1$.
- If we find $y \in S^\circ$ s.t. $y \cdot x > 1$, then $y$ is the separating hyperplane
  - $y^\intercal z \leq 1 < y^\intercal x$ for every $z \in S$.

# Optimization $\Longleftrightarrow$ Separation







# Beyond Polytopes

Essentially everything we proved about equivalence of separation and optimization for polytopes extends (approximately) to arbitrary convex sets.

# Beyond Polytopes

Essentially everything we proved about equivalence of separation and optimization for polytopes extends (approximately) to arbitrary convex sets.

Problems parametrized by $P$, a closed convex set.

## Weak Optimization Problem

- Input: Linear objective $c \in \mathbb{R}^n$.
- Output: $x \in P^{+\epsilon}$, and $c^\intercal x \geq \max_{x' \in P} c^\intercal x' - \epsilon$

## Weak Separation Problem

- Input: $y \in \mathbb{R}^n$
- Output: Decide that $y \in P^{-\epsilon}$, or else find $h \in \mathbb{R}^n$ with $||h|| = 1$ s.t. $h^\intercal x < h^\intercal y + \epsilon$ for all $x \in P$.

# Beyond Polytopes

Essentially everything we proved about equivalence of separation and optimization for polytopes extends (approximately) to arbitrary convex sets.

Problems parametrized by $P$, a closed convex set.

## Weak Optimization Problem

- Input: Linear objective $c \in \mathbb{R}^n$.
- Output: $x \in P^{+\epsilon}$, and $c^\intercal x \geq \max_{x' \in P} c^\intercal x' - \epsilon$

## Weak Separation Problem

- Input: $y \in \mathbb{R}^n$
- Output: Decide that $y \in P^{-\epsilon}$, or else find $h \in \mathbb{R}^n$ with $||h|| = 1$ s.t. $h^\intercal x < h^\intercal y + \epsilon$ for all $x \in P$.

I could have equivalently stated the weak optimization problem for convex functions instead of linear.

**Theorem (Equivalence of Separation and Optimization for Convex Sets)**

*Consider a family $\mathcal{P}$ of convex sets described implicitly using $\langle P \rangle$ bits. Then the weak separation problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle y \rangle, \log(1/\epsilon))$ time for $P \in \mathcal{P}$ if and only if the weak optimization problem is also solvable in $\mathrm{poly}(\langle P \rangle, n, \langle c \rangle, \log(1/\epsilon))$ time.*

## Theorem (Equivalence of Separation and Optimization for Convex Sets)

*Consider a family* $\mathcal{P}$ *of convex sets described implicitly using* $\langle P \rangle$ *bits. Then the weak separation problem is solvable in* $\text{poly}(\langle P \rangle, n, \langle y \rangle, \log(1/\epsilon))$ *time for* $P \in \mathcal{P}$ *if and only if the weak optimization problem is also solvable in* $\text{poly}(\langle P \rangle, n, \langle c \rangle, \log(1/\epsilon))$ *time.*

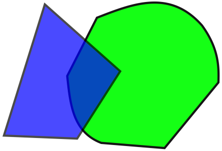
- The "approximation" in this statement is necessary, since we can't solve convex optimization problems exactly.
- Weak separation suffices for ellipsoid, which is only approximately optimal anyways
- By polarity, weak optimization is equivalent to weak separation

**Theorem (Equivalence of Separation and Optimization for Convex Sets)**

*Consider a family $\mathcal{P}$ of convex sets described implicitly using $\langle P \rangle$ bits. Then the weak separation problem is solvable in $\text{poly}(\langle P \rangle, n, \langle y \rangle, \log(1/\epsilon))$ time for $P \in \mathcal{P}$ if and only if the weak optimization problem is also solvable in $\text{poly}(\langle P \rangle, n, \langle c \rangle, \log(1/\epsilon))$ time.*

- The "approximation" in this statement is necessary, since we can't solve convex optimization problems exactly.
- Weak separation suffices for ellipsoid, which is only approximately optimal anyways
- By polarity, weak optimization is equivalent to weak separation
- For proof / details, see the GLS book.

# Implication: Operations preserving solvability

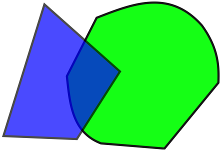


- Assume you can efficiently optimize over two convex sets $P$ and $Q$

**Question**

What about $P \bigcup Q$ and $P \bigcap Q$?

# Implication: Operations preserving solvability



- Assume you can efficiently optimize over two convex sets $P$ and $Q$

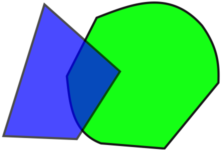
**Question**

What about $P \bigcup Q$ and $P \bigcap Q$?

**$P \bigcup Q$**

- Yes! Simply optimize over each separately, and take the better of the two outcomes.
- Equivalent to optimizing over the convex hull of $P \bigcup Q$.
- Implication of Separation/optimization equivalence: there is a separation oracle for $convexhull(P \bigcup Q)$.

# Implication: Operations preserving solvability



- Assume you can efficiently optimize over two convex sets $P$ and $Q$

## Question

What about $P \bigcup Q$ and $P \bigcap Q$?

## $P \bigcap Q$

- Yes! Follows from equivalence of separation and optimization.
- Specifically, can separate over $P$ and $Q$ individually, therefore can separate over $P \bigcap Q$, and then can optimize over $P \bigcap Q$.
- Applications: colorful spanning tree, cardinality-constrained matching, ...

# Implication: Constructive Caratheodory

## Problem

Given a point $x \in \mathcal{P}$, where $\mathcal{P} \subseteq \mathbb{R}^n$ is a solvable polytope, write $x$ as a convex combination of $n+1$ vertices of $\mathcal{P}$, and do so in polynomial time.

- Existence: Caratheodory's theorem.
- E.g. Birkhoff Von-Neumann, fractional spanning trees, fractional matchings, ...
- Follows from equivalence of separation and optimization. See HW.