CS675: Convex and Combinatorial Optimization
Spring 2018
Introduction to Matroid Theory

Instructor: Shaddin Dughmi

Optimization over Sets

- Most combinatorial optimization problems can be thought of as choosing the best set from a family of allowable sets
  - Shortest paths
  - Max-weight matching
  - Independent set
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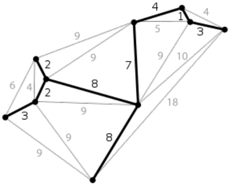
- Analogues of concave and convex: submodular and supermodular (in no particular order!)
- Today, we will look only at optimizing modular objectives over an extremely prolific family of set systems
  - Related, directly or indirectly, to a large fraction of optimization problems in P
  - Also pops up in submodular/supermodular optimization problems

Outline

Maximum Weight Forest Problem



Given a connected undirected graph $G = (V, E)$, and weights $w_e \in \mathbb{R}$ on edges e, find a maximum weight acyclic subgraph (aka forest) of G.

- Slight generalization of minimum weight spanning tree
- We use n and m to denote $|V|$ and $|E|$, respectively.

The Greedy Algorithm

1. $B \leftarrow \emptyset$
2. Sort non-negative weight edges in decreasing order of weight
   - $e_1, \ldots, e_m$, with $w_1 \geq w_2 \geq \ldots \geq w_m \geq 0$
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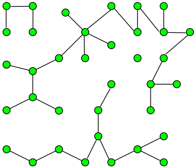
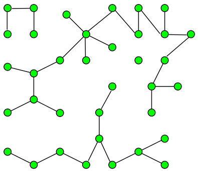
Theorem

The greedy algorithm outputs a maximum-weight forest.



Lemma

1. The empty set is acyclic
2. If A is an acyclic set of edges, and $B \subseteq A$, then B is also acyclic.
3. If A, B are acyclic, and $|B| > |A|$, then there is $e \in B \setminus A$ such that $A \bigcup \{e\}$ is acyclic



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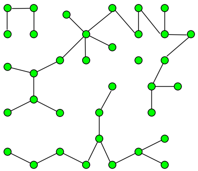
(1) and (2) are trivial, so let's prove (3)



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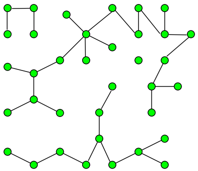

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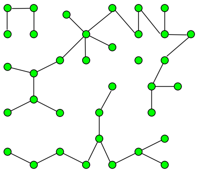

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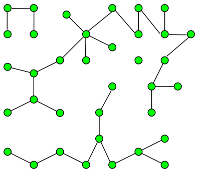

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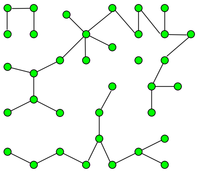

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1. The empty set is acyclic
2. If A is an acyclic set of edges, and $B \subseteq A$, then B is also acyclic.
   - Contrapositive: if B cyclic then so is A
3. If A, B are acyclic, and $|B| > |A|$, then there is $e \in B \setminus A$ such that $A \bigcup \{e\}$ is acyclic
   - Inductively: can extend A by adding $|B| - |A|$ elements from $B \setminus A$

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Proof

Going back to proving the algorithm correct.

Inductive Hypothesis (i)

There is a maximum-weight acyclic forest B_i^* which "agrees" with the algorithm's choices on edges $e_1, \ldots, e_i$.

- i.e. if B_i denotes the algorithm's choice up to iteration i, then $B_i = B_i^* \bigcap \{e_1, \ldots, e_i\}$

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- If $e_i \notin B_i$, then $B_{i-1} \bigcup \{e_i\}$ is cyclic. Since $B_{i-1} \subseteq B_{i-1}^*$, then $e_i \notin B_{i-1}^*$ (Property 2). So take $B_i^* = B_{i-1}^*$.

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- If $e_i \in B_i$ and $e_i \notin B_{i-1}^*$, build B_i^* by repeatedly extending B_i using B_{i-1}^* (property 3)
  - Recall that $B_i = B_{i-1} \bigcup \{e_i\}$ agrees with B_{i-1}^* on $e_1, \ldots, e_{i-1}$.
  - $B_i^* = B_{i-1}^* \bigcup \{e_i\} \setminus \{e_k\}$ for some $k > i$
  - B_i^* has weight no less than B_{i-1}^*, so optimal.

To prove optimality of the greedy algorithm, all we needed was the following.

Matroids

A set system $M = (\mathcal{X}, \mathcal{I})$ is a matroid if

1. $\emptyset \in \mathcal{I}$
2. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (Downward Closure)
3. If $A, B \in \mathcal{I}$ and $|B| > |A|$, then $\exists\, x \in B \setminus A$ such that $A \bigcup \{x\} \in \mathcal{I}$ (Exchange Property)

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- The matroid whose independent sets are acyclic subgraphs is called a graphic matroid
- Other examples abound!

Example: Linear Matroid

- $\mathcal{X}$ is a finite set of vectors $\{v_1, \ldots, v_m\} \subseteq \mathbb{R}^n$
- $S \in \mathcal{I}$ iff the vectors in S are linearly independent

- Downward closure: If a set of vectors is linearly independent, then every subset of it is also
- Exchange property: Can always extend a low-dimension independent set S by adding vectors from a higher dimension independent set T

Example: Uniform Matroid

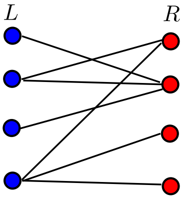
- $\mathcal{X}$ is an arbitrary finite set $\{1, \ldots, n\}$.
- $S \in \mathcal{I}$ iff $|S| \leq k$.

- Downward closure: If a set S has $|S| \leq k$ then the same holds for $T \subseteq S$.
- Exchange property: If $|S| < |T| \leq k$, then there is an element in $T \setminus S$, and we can add it to S while preserving independence.

Example: Partition Matroid

- $\mathcal{X}$ is the disjoint union of classes $X_1, \ldots, X_m$
- Each class X_j has an upperbound k_j.
- $S \in \mathcal{I}$ iff $|S \bigcap X_j| \leq k_j$ for all j

- This is the “disjoint union” of a number of uniform matroids



Example: Transversal Matroid

- Described by a bipartite graph $E \subseteq L \times R$
- $\mathcal{X} = L$
- $S \in \mathcal{I}$ iff there is a bipartite matching which matches S

- Downward closure: If we can match S, then we can match $T \subseteq S$.
- Exchange property: If $|T| > |S|$ is matchable, then an augmenting path/alternating path extends the matching of S to some $x \in T \setminus S$.

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3. For $i = 1$ to n:
   - if $B \bigcup \{i\} \in \mathcal{I}$, add i to B.

Theorem

The greedy algorithm returns the maximum weight set for every choice of weights if and only if the set system $(\mathcal{X}, \mathcal{I})$ is a matroid.

We already saw the "if" direction. We will skip "only if".

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  - A subroutine which checks whether $S \in \mathcal{I}$ or not.
- Runs in time $O(n \log n) + nT$, where T is runtime of the independence oracle.

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- For most "natural" matroids, independence oracle is easy to implement efficiently
  - Graphic matroid
  - Linear matroid
  - Uniform/partition matroid
  - Transversal matroid

Outline

Independent Sets, Bases, and Circuits

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What are these for:

- Graphic matroid
- Linear matroid
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Lemma

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The following analogue of vector space dimension is well-defined.

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- The Rank of $S \subseteq \mathcal{X}$ in $\mathcal{M}$ is the size of the maximal independent subsets (i.e. bases) of S.
- The rank of $\mathcal{M}$ is the size of the bases of $\mathcal{M}$.
- The function $rank_{\mathcal{M}}(S) : 2^{\mathcal{X}} \rightarrow \mathbb{N}$ is called the rank function of $\mathcal{M}$.

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E.g.: Graphic matroid, linear matroid, transversal matroid

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Observation

$i \in \{1, \ldots, n\}$ is selected by the greedy algorithm iff
$i \notin span(\{1, \ldots, i-1\})$

Operations preserving Matroidness

Given $\mathcal{M} = (\mathcal{X}, \mathcal{I})$, consider the following operations:

- Deletion: For $B \subseteq \mathcal{X}$, we define $\mathcal{M} \setminus B = (\mathcal{X}', \mathcal{I}')$ with $\mathcal{X}' = \mathcal{X} \setminus B$,

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- Contraction: Given $B \subseteq \mathcal{X}$, let $M/B = (\mathcal{X}', \mathcal{I}')$ with $\mathcal{X}' = \mathcal{X} \setminus B$,
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- Others: truncation, dual, union...

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- Optimization over matroids is “easy”, in the same way that optimization over convex sets is “easy”
- Operations preserving set convexity are analogous to operations preserving matroid structure

Matroids as an Algebra of Tractable Discrete Problems

- Optimization over matroids is “easy”, in the same way that optimization over convex sets is “easy”
- Operations preserving set convexity are analogous to operations preserving matroid structure
- Arguably, matroids and submodular functions are discrete analogues of convex sets and convex functions, respectively.
    - Less exhaustive

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  - The polytope is “solvable”, and admits a polytime separation oracle
- This perspective will be crucial for more advanced applications of matroids
  - Optimization of linear functions over matroid intersections
  - Optimization of submodular functions over matroids

The Matroid Polytope

Polytope $\mathcal{P}(\mathcal{M})$ for $\mathcal{M} = (\mathcal{X}, \mathcal{I})$

$$\sum_{i \in S} x_i \leq rank_{\mathcal{M}}(S), \quad \text{for } S \subseteq \mathcal{X}.$$
$$x_i \geq 0, \quad \text{for } i \in \mathcal{X}.$$

- Assigns a variable x_i to every element i of the ground set
- Each feasible x is a fractional subset of $\mathcal{X}$
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- Note: polytope has $2^{|\mathcal{X}|}$ constraints.

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- Recall: suffices to show that every linear function $w^T x$ is maximized over $\mathcal{P}(\mathcal{M})$ at some x_I for $I \in \mathcal{I}$.

Recall:The Greedy Algorithm

1. $B \leftarrow \emptyset$
2. Sort nonnegative elements of $\mathcal{X}$ in decreasing order of weight
   - $\{1, \ldots, n\}$ with $w_1 \geq w_2, \geq \ldots \geq w_n \geq 0$.
3. For $i = 1$ to n:
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- We can think of the greedy algorithm as computing the indicator vector $x^* = x_B \in \mathcal{P}(\mathcal{M})$
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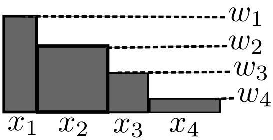

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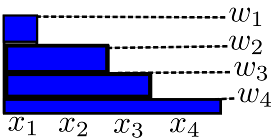

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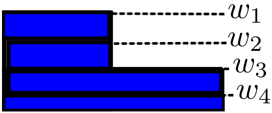

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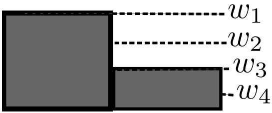

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  - Graphic: convex hull of forests
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- Since $||x||_1 = rank(\mathcal{M})$, and $||x_{I_\ell}||_1 \leq rank(\mathcal{M})$ for all ℓ, it must be that $||x_{I_1}||_1 = ||x_{I_2}||_1 = \ldots = ||x_{I_k}||_1 = rank(\mathcal{M})$

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- Therefore, by equivalence of separation and optimization, can also implement a separation oracle for $\mathcal{P}(\mathcal{M})$ in $\mathrm{poly}(n, T)$ time.
- A more direct proof: reduces to submodular function minimization
  - $rank_{\mathcal{M}}$ is a submodular set function.

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Matroid Intersection

- Optimization of linear functions over matroids is tractable
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- However, maximizing linear functions over the intersection of 3 or more matroids is NP-hard

Examples

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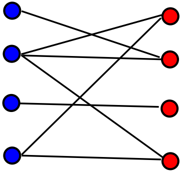

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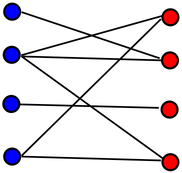
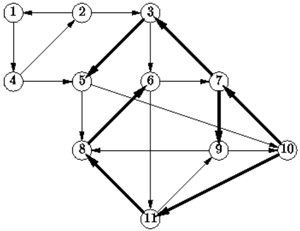


Arborescence

Given a directed graph G, a set of edges is an r-arborescence if it is a tree directed away from the root r.



Examples

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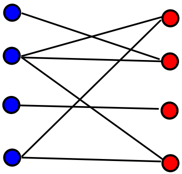
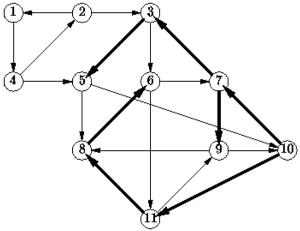
Given a bipartite graph G, a set of edges F is a bipartite matching if and only if each node is incident on at most one edge in F.



Arborescence

Given a directed graph G, a set of edges is an r-arborescence if it is a tree directed away from the root r.



- Others: colorful spanning trees, orientations, . . .

The Matroid Intersection Polytope

Matroid Intersection

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- Optimizing a modular function over $\mathcal{M}_1 \bigcap \mathcal{M}_2$ is equivalent to optimizing a linear function over $convexhull\,\{x_I : I \in \mathcal{I}_1 \bigcap \mathcal{I}_2\}$.
- As it turns out, this is a solvable polytope.

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- Nevertheless, it is true but hard to prove, so we will skip it.

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Theorem

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Proof: Using equivalence of separation and optimization, and the fact that all coefficients in the LP have $\mathrm{poly}(n)$ bits.

NP-hardness of 3-way Matroid Intersection

By a reduction from Hamiltonian Path in directed graphs