CS675: Convex and Combinatorial Optimization
Spring 2022
Combinatorial Problems as Linear and Convex Programs

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Outline

Combinatorial vs Convex Optimization

- In CS, discrete problems are traditionally viewed/analyzed using discrete mathematics and combinatorics
  - Algorithms are combinatorial in nature (greedy, dynamic programming, divide and conquor, etc)

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  - Usually linear programs, but increasingly more general convex programs

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  - Algorithms are combinatorial in nature (greedy, dynamic programming, divide and conquor, etc)
- In OR and optimization community, these problems are often expressed as continuous optimization problems
  - Usually linear programs, but increasingly more general convex programs
- Increasingly in recent history, it is becoming clear that combining both viewpoints is the way to go
  - Better algorithms (runtime, approximation)
  - Structural insights (e.g. market clearing prices in matching markets)
  - Unifying theories and general results (Matroids, submodular optimization, constraint satisfaction)

Discrete Problems as Linear Programs

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  - Dantzig's original application was the problem of matching 70 people to 70 jobs!
- This is not surprising, since almost any finite family of discrete objects can be encoded as a finite subset of Euclidean space
  - Convex hull of that set is a polytope
  - E.g. spanning trees, paths, cuts, TSP tours, assignments...

Discrete Problems as Linear Programs

- LP algorithms typically require representation as a “small” family of inequalities,
  - Not possible in general (Say when problem is NP-hard, assuming $(P \neq NP)$)
  - Shown unconditionally impossible in some cases (e.g. TSP)

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- But, in many cases, polyhedra in inequality form can be shown to encode a combinatorial problems at the vertices

Next

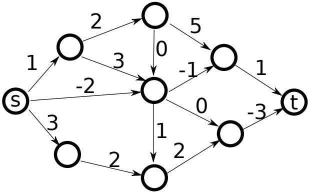
We examine some combinatorial problems through the lense of LP and convex optimization, starting with shortest path.

Outline

The Shortest Path Problem

Given a directed graph $G = (V, E)$ with cost $c_e \in \mathbb{R}$ on edge e, find the minimum cost path from s to t.

- We use n and m to denote $|V|$ and $|E|$, respectively.
- We allow costs to be negative, but assume no negative cycles
- We assume that there is some path from s to t (Check via BFS)



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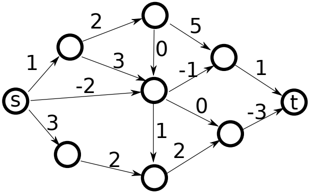

When costs are nonnegative, Dijkstra's algorithm finds the shortest path from s to every other node in time $O(m + n \log n)$.

Using primal/dual paradigm, we will design a polynomial-time algorithm that works when graph has negative edges but no negative cycles

Note: Negative Edges and Complexity

- When the graph has no negative cycles, there is a shortest path which is simple
- When the graph has negative cycles, there may not be a shortest path from s to t.
- In these cases, the algorithm we design can be modified to "fail gracefully" by detecting such a cycle
  - Can be used to detect arbitrage opportunities in currency exchange networks

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- In these cases, the algorithm we design can be modified to "fail gracefully" by detecting such a cycle
  - Can be used to detect arbitrage opportunities in currency exchange networks
- In the presence of negative cycles, finding the shortest simple path is NP-hard (by reduction from Hamiltonian cycle)

An LP Relaxation of Shortest Path

Consider the following LP

Primal Shortest Path LP

$$
\begin{aligned}
&\text{min } \textstyle\sum_{e \in E} c_e x_e \\
&\text{s.t.} \\
&\sum_{e \to v} x_e - \sum_{v \to e} x_e = \delta_v, \quad \forall v \in V. \\
&x_e \geq 0, \qquad\qquad\qquad\qquad \forall e \in E.
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where $\delta_v = -1$ if $v = s$, 1 if $v = t$, and 0 otherwise.

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- This is a relaxation of the shortest path problem
  - Indicator vector x_P of $s - t$ path P is a feasible solution, with cost as given by the objective
    - LP is feasible
  - Fractional feasible solutions may not correspond to paths

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    - LP is feasible
  - Fractional feasible solutions may not correspond to paths
- A-priori, it is conceivable that optimal value of LP is less than length of shortest path.

Integrality of the Shortest Path Polyhedron

$$
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$$

We will show that above LP encodes the shortest path problem exactly

Claim

When c satisfies the no-negative-cycles property, the indicator vector of the shortest $s - t$ path is an optimal solution to the LP.

Dual LP

We will use the following LP dual

Primal LP

min $\sum_{e \in E} c_e x_e$
s.t.

$$\sum_{e \to v} x_e - \sum_{v \to e} x_e = \delta_v, \quad \forall v \in V.$$
$$x_e \geq 0, \quad \forall e \in E.$$

Dual LP

max $y_t - y_s$
s.t.

$$y_v - y_u \leq c_e, \quad \forall (u, v) \in E.$$

- Interpretation of dual variables y_v: "height" or "potential"
- Relative potential of vertices constrained by length of edge between them (triangle inequality)
- Dual is trying to maximize relative potential of s and t,

Proof Using the Dual

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- Let x^* be indicator vector of shortest s-t path
  - Feasible for primal
- Let y_v^* be shortest path distance from s to v
  - Feasible for dual (by triangle inequality)
- $\sum_e c_e x_e^* = y_t^* - y_s^*$, so both x^* and y^* optimal.

Integrality of Polyhedra

A stronger statement is true:

Integrality of Shortest Path LP

The vertices of the polyhedral feasible region are precisely the indicator vectors of simple paths in G.

- Implies that there always exists a vertex optimal solution which is a path whenever LP is bounded
  - We will also show that LP is bounded precisely when c has no negative cycles.
- Reduces computing shortest path in graphs with no negative cycles to finding optimal vertex of LP

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2. Fact: For every LP vertex x there is objective c such that x is unique optimal. (Prove it!)

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2. Fact: For every LP vertex x there is objective c such that x is unique optimal. (Prove it!)
3. Since such a c satisfies no-negative-cycles property, claim on previous slide shows that x is integral.

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In general, the approach we took applies in many contexts: To show a polytope's vertices integral, it suffices to show that there is an integral optimal for any objective which admits an optimal solution.

Outline

Ford's Algorithm

Primal LP

min $\sum_{e \in E} c_e x_e$
s.t.

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Dual LP

max $y_t - y_s$
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$$y_v - y_u \leq c_e, \quad \forall e = (u, v) \in E.$$

For convenience, add (s, v) of length ∞ when one doesn't exist.

Ford's Algorithm

1. $y_v = c_{(s,v)}$ and $pred(v) = s$ for $v \neq s$
2. $y_s = 0$, $pred(s) = null$.
3. While some dual constraint is violated, i.e. $y_v > y_u + c_e$ for some $e = (u, v)$
   - Set $pred(v) = u$ (To get from s to v, take shortcut through u)
   - Set $y_v = y_u + c_e$
4. Output the path $t, pred(t), pred(pred(t)), \ldots, s$.

Correctness

Lemma (Loop Invariant 1)

Assuming no negative cycles, $pred$ defines a path P from s to t, of length at most $y_t - y_s$. (Hence also $y_t - y_s \geq distance(s, t)$)

Interpretation

- Ford's algorithm maintains an (initially infeasible) dual y
- Also maintains feasible primal P of length $\leq$ dual objective $y_t - y_s$
- Iteratively "fixes" dual y, tending towards feasibility
- Once y is feasible, weak duality implies P optimal.

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Algorithms of this form, that output a matching primal and dual solution, are called Primal-Dual Algorithms.

Termination

Lemma (Loop Invariant 2)

Assuming no negative cycles, y_v is the length of some simple path from s to v.

Termination

Lemma (Loop Invariant 2)

Assuming no negative cycles, y_v is the length of some simple path from s to v.

Theorem (Termination)

When the graph has no negative cycles, Ford's algorithm terminates in a finite number of steps.

Proof

- The graph has a finite number N of simple paths
- By loop invariant 2, every dual variable y_v is the length of some simple path.
- Dual variables are nonincreasing throughout algorithm, and one decreases each iteration.
- There can be at most nN iterations.

Observation: Single source shortest paths

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Observation

Algorithm does not depend on t till very last step. So essentially solves the single-source shortest path problem. i.e. finds shortest paths from s to all other vertices v.

Loop Invariant 1

We prove Loop Invariant 1 through two Lemmas

Lemma (Loop Invariant 1a)

For every node w, we have $y_w - y_{pred(w)} \geq c_{pred(w),w}$

Proof

- Fix w
- Holds at first iteration
- Preserved by Induction on iterations
  - If neither y_w nor $y_{pred(w)}$ updated, nothing changes.
  - If y_w (and $pred(w)$) updated, then $y_w = y_{pred(w)} + c_{pred(w),w}$
  - $y_{pred(w)}$ updated, it only goes down, preserving inequality.

Loop Invariant 1

Lemma (Invariant 1b)

Assuming no negative cycles, pred forms a directed tree rooted out of s.

We denote this path from s to a node w by $P(s,w)$.

Proof

- Holds at first iteration
- For a contradiction, consider iteration of first violation
  - v and u with $y_v > y_u + c_{u,v}$
- $P(s,u)$ passes through v
  - Otherwise tree property preserved by setting $pred(v) = u$
- Let $P(v,u)$ be the portion of $P(s,u)$ starting at v.
- By Invariant 1a, and telescoping sum, length of $P(v,u)$ is at most $y_u - y_v$.
- Length of cycle $\{P(v,u),(u,v)\}$ at most $y_u - y_v + c_{u,v} < 0$.

Summarizing Loop Invariant 1

Lemma (Invariant 1a)

For every node w, we have $y_w - y_{pred(w)} \geq c_{pred(w),w}$.

- By telescoping sum, can bound $y_w - y_s$ when pred leads back to s

Lemma (Invariant 1b)

Assuming no negative cycles, pred forms a directed tree rooted out of s.

- Implies that following $pred$ always leads back to s, and that y_s remains 0.

Corollary (Loop Invariant 1)

Assuming no negative cycles, $pred$ defines a path $P(s, w)$ from s to each node w, of length at most $y_w - y_s = y_w$. (Hence $y_w \geq distance(s, w)$)

Loop Invariant 2

Lemma (Loop Invariant 2)

Assuming no negative cycles, y_w is the length of some simple path $Q(s, w)$ from s to w, for all w.

Proof is technical, by induction, so we will skip. Instead, we will modify Ford's algorithm to guarantee polynomial time termination.

Bellman-Ford Algorithm

The following algorithm fixes an (arbitrary) order on edges E

Bellman-Ford Algorithm

1. $y_v = c_{(s,v)}$ and $pred(v) = s$ for $v \neq s$
2. $y_s = 0$, $pred(s) = null$.
3. While y is infeasible for the dual
   - For $e = (u, v)$ in order, if $y_v > y_u + c_e$ then
     - Set $pred(v) = u$ (To get from s to v, take shortcut through u)
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Note

Correctness follows from the correctness of Ford's Algorithm.

Runtime

Theorem

Bellman-Ford terminates after $n - 1$ scans through E, for a total runtime of $O(nm)$.

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Follows immediately from the following Lemma

Lemma

After k scans through E, vertices v with a shortest $s-v$ path consisting of $\leq k$ edges are correctly labeled. (i.e., $y_v = distance(s, v)$)

Proof

Lemma

After k scans through E, vertices v with a shortest $s - v$ path consisting of $\leq k$ edges are correctly labeled. (i.e., $y_v = distance(s, v)$)

Proof

- Holds for $k = 0$
- By induction on k.
  - Assume it holds for $k - 1$.
  - Let v be a node with a shortest path P from s with k edges.
  - $P = \{Q, e\}$, for some $e = (u, v)$ and $s - u$ path Q, where Q is a shortest $s - u$ path and Q has $k - 1$ edges.
  - By inductive hypothesis, u is correctly labeled before e is scanned for kth time – i.e. $y_u = distance(s, u)$.
  - Therefore, v is correctly labeled $y_v = y_u + c_{u,v} = distance(s, v)$ after e is scanned for kth time

A Note on Negative Cycles

Question

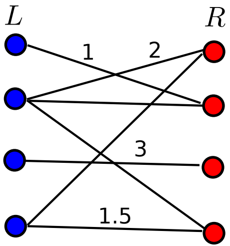
What if there are negative cycles? What does that say about LP? What about Ford's algorithm?

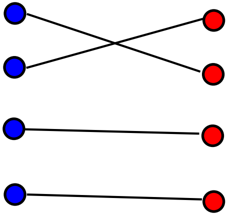
Outline

The Max-Weight Bipartite Matching Problem

Given a bipartite graph $G = (V, E)$, with $V = L \bigcup R$, and weights w_e on edges e, find a maximum weight matching.

- Matching: a set of edges covering each node at most once
- We use n and m to denote $|V|$ and $|E|$, respectively.
- Equivalent to maximum weight / minimum cost perfect matching.





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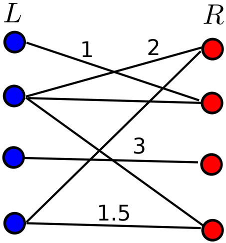

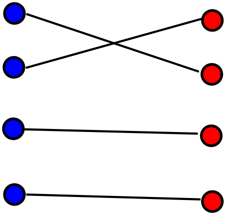

Our focus will be less on algorithms, and more on using polyhedral interpretation to gain insights about a combinatorial problem.

An LP Relaxation of Bipartite Matching

Bipartite Matching LP

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- Feasible region is a polytope $\mathcal{P}$ (i.e. a bounded polyhedron)
- This is a relaxation of the bipartite matching problem
  - Integer points in $\mathcal{P}$ are the indicator vectors of matchings.

$$\mathcal{P} \cap \mathbb{Z}^m = \{x_M : M \text{ is a matching}\}$$

Integrality of the Bipartite Matching Polytope

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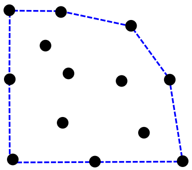
$$x_e \geq 0, \qquad \forall e \in E.$$



Theorem

The feasible region of the matching LP is the convex hull of indicator vectors of matchings.

$$\mathcal{P} = \text{convexhull}\{x_M : M \text{ is a matching}\}$$

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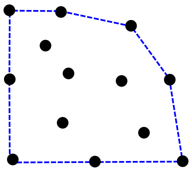

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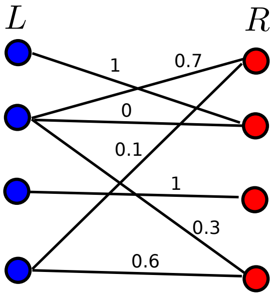
$$\mathcal{P} = \text{convexhull}\left\{x_M : M \text{ is a matching}\right\}$$

Note

- This is the strongest guarantee you could hope for of an LP relaxation of a combinatorial problem
- Solving LP is equivalent to solving the combinatorial problem
- Stronger guarantee than shortest path LP from last time

Proof



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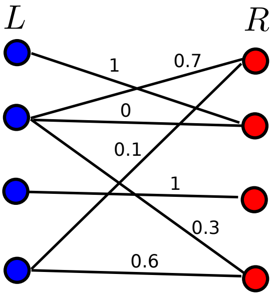

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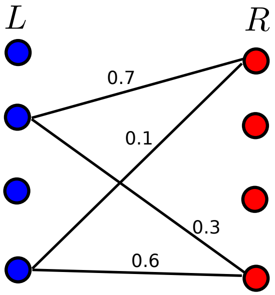

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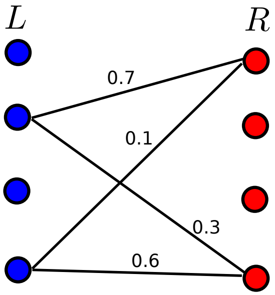

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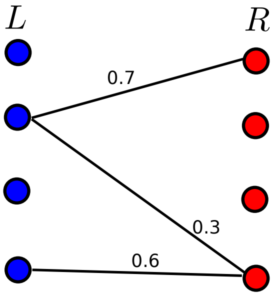

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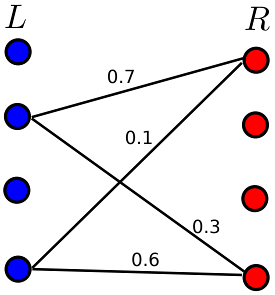
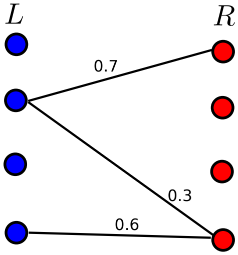


Case 1: Cycle C

- Let $C = (e_1, \ldots, e_k)$, with k even
- There is $\epsilon > 0$ such that adding $\pm\epsilon(+1, -1, \ldots, +1, -1)$ to x_C preserves feasibility
- x is the midpoint of $x + \epsilon(+1, -1, ..., +1, -1)_C$ and $x - \epsilon(+1, -1, \ldots, +1, -1)_C$, so x is not a vertex.

Proof



Case 2: Maximal Path P

- Let $P = (e_1, \ldots, e_k)$, going through vertices $v_0, v_1, \ldots, v_k$
- By maximality, e_1 is the only edge of v_0 with non-zero x-weight
  - Similarly for e_k and v_k.
- There is $\epsilon > 0$ such that adding $\pm\epsilon(+1, -1, \ldots, ?1)$ to x_P preserves feasibility
- x is the midpoint of $x + \epsilon(+1, -1, ..., ?1)_P$ and $x - \epsilon(+1, -1, \ldots, ?1)_P$, so x is not a vertex.

Related Fact: Birkhoff Von-Neumann Theorem

$$\begin{aligned} \sum_{e \in \delta(v)} x_e &= 1, \quad && \forall v \in V. \\ x_e &\geq 0, && \forall e \in E. \end{aligned}$$

- The analogous statement holds for the perfect matching LP above, by an essentially identical proof.

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The set of $n \times n$ doubly stochastic matrices is the convex hull of $n \times n$ permutation matrices.

$$\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} = 0.5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.5 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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By Caratheodory's theorem, we can express every doubly stochastic matrix as a convex combination of $n^2 + 1$ permutation matrices.

We will see later: this decomposition can be computed efficiently!

Outline

Total Unimodularity

We could have proved integrality of the bipartite matching LP using a more general tool

Definition

A matrix A is Totally Unimodular if every square submatrix has determinant 0, $+1$ or -1.

Theorem

If $A \in \mathbb{R}^{m \times n}$ is totally unimodular, and b is an integer vector, then $\{x : Ax \leq b, x \geq 0\}$ has integer vertices.

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Proof

- Non-zero entries of vertex x are solution of $A'x' = b'$ for some nonsingular square submatrix A' and corresponding sub-vector b'
- Cramer's rule:

$$x_i' = \frac{\det(A_i' | b')}{\det A'}$$

Total Unimodularity of Bipartite Matching

$$\sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V.$$

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- If A' has all-zero column, then $\det A' = 0$
- If A' has column with single 1, then holds by induction.
- If all columns of A' have two 1's,
  - Partition rows (vertices) into L and R
  - Sum of rows L is $(1, 1, \ldots, 1)$, similarly for R
  - A' is singular, so $\det A' = 0$.

Outline

Primal and Dual LPs

Primal LP

$$\max \sum_{e \in E} w_e x_e$$
s.t.
$$\sum_{e \in \delta(v)} x_e \leq 1, \qquad \forall v \in V.$$
$$x_e \geq 0, \qquad \forall e \in E.$$

Dual LP

$$\min \sum_{v \in V} y_v$$
s.t.
$$y_u + y_v \geq w_e, \quad \forall e = (u, v) \in E.$$
$$y_v \succeq 0, \qquad \forall v \in V.$$

- Primal interpertation: Player 1 looking to build a set of projects
  - Each edge e is a project generating “profit” w_e
  - Each project $e = (u, v)$ needs two resources, u and v
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- Dual interpretation: Player 2 looking to buy resources
  - Offer a price y_v for each resource.
  - Prices should incentivize player 1 to sell resources
  - Want to pay as little as possible.

Vertex Cover Interpretation

Primal LP

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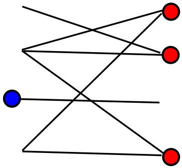
When edge weights are 1, binary solutions to dual are vertex covers

Definition

$C \subseteq V$ is a vertex cover if every $e \in E$ has at least one endpoint in C



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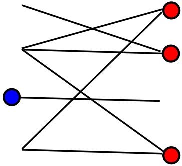

- Dual is a relaxation of the minimum vertex cover problem for bipartite graphs.
- By weak duality: min-vertex-cover $\geq$ max-cardinality-matching

König's Theorem

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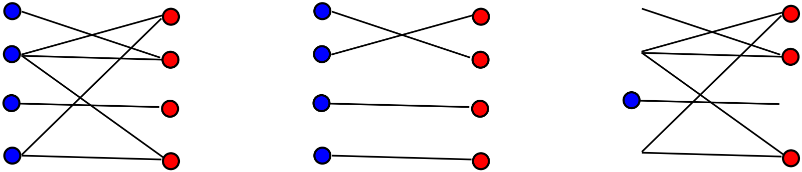
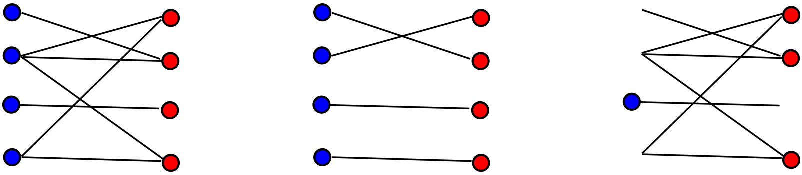
König's Theorem

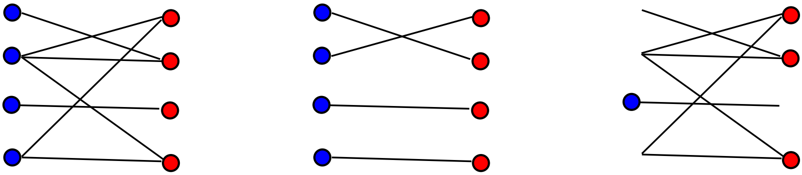
In a bipartite graph, the cardinality of the maximum matching is equal to the cardinality of the minimum vertex cover.

i.e. the dual LP has an integral optimal solution

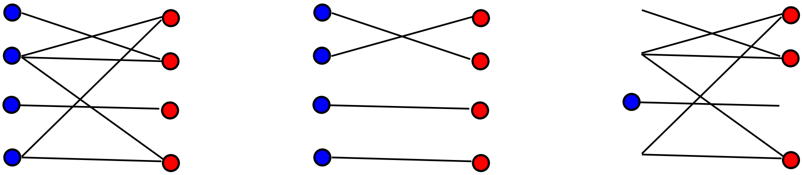




- Let $M(G)$ be a max cardinality of a matching in G
- Let $C(G)$ be min cardinality of a vertex cover in G
- We already proved that $M(G) \leq C(G)$
- We will prove $C(G) \leq M(G)$ by induction on number of nodes in G.



- Let y be an optimal dual, and v a vertex with $y_v > 0$



- Let y be an optimal dual, and v a vertex with $y_v > 0$
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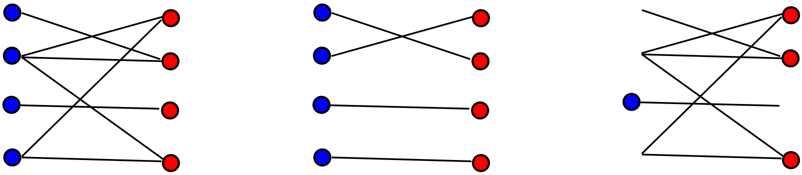

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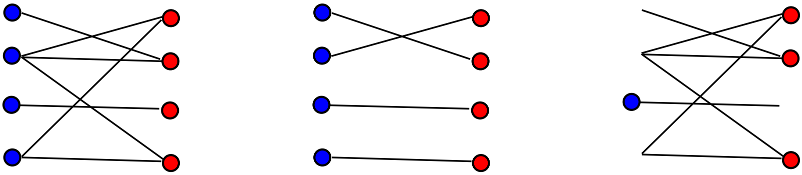

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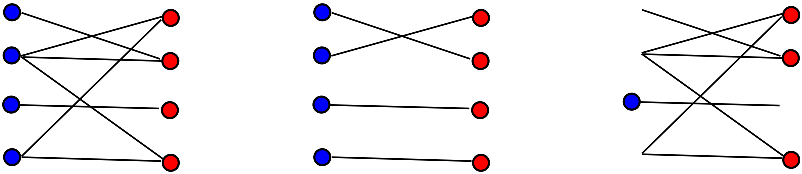

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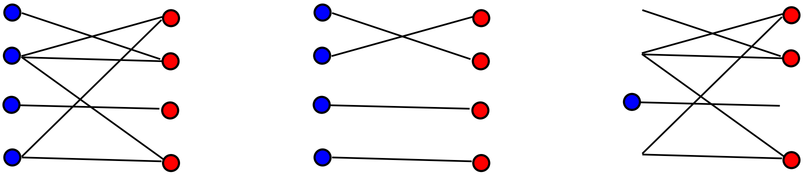

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Note: Could have proved the same using total unimodularity

Consequences of König's Theorem

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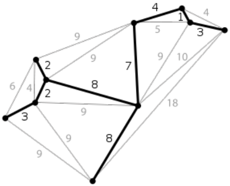
- Vertex covers can serve as a certificate of optimality for bipartite matchings, and vice versa
- Like maximum cardinality matching, minimum cardinality vertex cover in bipartite graphs can be formulated as an LP, and solved in polynomial time
- The same is true for the maximum independent set problem in bipartite graphs.
  - C is a vertex cover iff $V \setminus C$ is an independent set.

Outline

The Minimum Cost Spanning Tree Problem



Given a connected undirected graph $G = (V, E)$, and costs c_e on edges e, find a minimum cost spanning tree of G.

- Spanning Tree: an acyclic set of edges connecting every pair of nodes
- When graph is disconnected, can search for min-cost spanning forest instead
- We use n and m to denote $|V|$ and $|E|$, respectively.

Kruskal's Algorithm

The minimum spanning tree problem can be solved efficiently by a simple greedy algorithm

Kruskal's algorithm

1. $T = \emptyset$
2. Sort edges in increasing order of cost
3. For each edge e in order
    - if $T \bigcup e$ is acyclic, add e to T.

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- Proof of correctness is via a simple exchange argument.
- Generalizes to Matroids

MST Linear Program

MST LP

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\text{minimize} & \sum_{e \in E} c_e x_e & \\
\text{subject to} & \displaystyle\sum_{e \in E} x_e = n - 1 & \\
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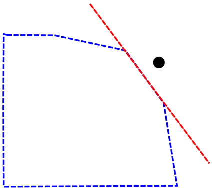
- Note: this LP has an exponential (in n) number of constraints

Solving the MST Linear Program



Definition

A separation oracle for a linear program with feasible set $\mathcal{P} \subseteq \mathbb{R}^m$ is an algorithm which takes as input $x \in \mathbb{R}^m$, and either certifies that $x \in \mathcal{P}$ or identifies a violated constraint.

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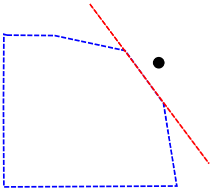

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Theorem

A linear program with a polynomial number of variables is solvable in polynomial time if and only if it admits a polynomial time separation oracle (modulo some technicalities)

Follows from the ellipsoid method, which we will see next week.

Solving the MST Linear Program

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- Given $x \in \mathbb{R}^m$, separation oracle must find a violated constraint if one exists
- Reduces to finding nonempty $X \subset V$ with $\sum_{e \subseteq X} x_e > |X| - 1$, if one exists
  - Equivalently $|X| - \sum_{e \subseteq X} x_e < 1$

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\text{subject to} & \sum\limits_{e \subseteq X} x_e \leq |X| - 1, & \text{for nonempty } X \subset V. \\
& \sum\limits_{e \in E} x_e = n - 1 & \\
& x_e \geq 0, & \text{for } e \in E.
\end{array}
$$

- Given $x \in \mathbb{R}^m$, separation oracle must find a violated constraint if one exists
- Reduces to finding nonempty $X \subset V$ with $\sum_{e \subseteq X} x_e > |X| - 1$, if one exists
  - Equivalently $|X| - \sum_{e \subseteq X} x_e < 1$
- In turn, this reduces to minimizing $|X| - \sum_{e \subseteq X} x_e$ over $X \subset V$

Solving the MST Linear Program

Primal LP

$$
\begin{array}{lll}
\text{minimize} & \sum_{e \in E} c_e x_e & \\
\text{subject to} & \sum_{e \subseteq X} x_e \leq |X| - 1, & \text{for nonempty } X \subset V. \\
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- In turn, this reduces to minimizing $|X| - \sum_{e \subseteq X} x_e$ over $X \subset V$

We will see how to do this efficiently later in the class, using submodular minimization

Application of Fractional Spanning Trees

- The LP formulation of spanning trees has many applications
- We will look at one contrived yet simple application that shows the flexibility enabled by polyhedral formulation

Fault-Tolerant MST

- Your tree is an overlay network on the internet used to transmit data
- A hacker is looking to attack your tree, by knocking off one of the edges of the graph
- You can foil the hacker by choosing a random tree
- The hacker knows the algorithm you use, but not your random coins

Fault-tolerant MST LP

$$\begin{array}{lll} \text{minimize} & \sum_{e \in E} c_e x_e & \\ \text{subject to} & \displaystyle\sum_{e \subseteq X} x_e \leq |X| - 1, & \text{for } X \subset V. \\ & \displaystyle\sum_{e \in E} x_e = n - 1 & \\ & x_e \leq p, & \text{for } e \in E. \\ & x_e \geq 0, & \text{for } e \in E. \end{array}$$

- Above LP can be solved efficiently
- If feasible, can interpret resulting fractional spanning tree x as a recipe for a probability distribution over trees T
  - $e \in T$ with probability x_e
  - Since $x_e \leq p$, no edge is in the tree with probability more than p.

Fault-tolerant MST LP

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\begin{array}{lll}
\text{minimize} & \sum_{e \in E} c_e x_e & \\
\text{subject to} & \displaystyle\sum_{e \subseteq X} x_e \le |X| - 1, & \text{for } X \subset V. \\
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& x_e \le p, & \text{for } e \in E. \\
& x_e \ge 0, & \text{for } e \in E.
\end{array}
$$

- Given feasible solution x, such a probability distribution exists!

Fault-tolerant MST LP

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\begin{array}{lll}
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\end{array}
$$

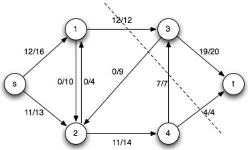
- Given feasible solution x, such a probability distribution exists!
  - x is in the (original) MST polytope
  - Caratheodory's theorem: x is a convex combination of $m + 1$ vertices of MST polytope
  - By integrality of MST polytope: x is the “expectation” of a probability distribution over spanning trees.

Fault-tolerant MST LP

$$\begin{array}{lll} \text{minimize} & \sum_{e \in E} c_e x_e & \\ \text{subject to} & \displaystyle\sum_{e \subseteq X} x_e \leq |X| - 1, & \text{for } X \subset V. \\ & \displaystyle\sum_{e \in E} x_e = n - 1 & \\ & x_e \leq p, & \text{for } e \in E. \\ & x_e \geq 0, & \text{for } e \in E. \end{array}$$

- Given feasible solution x, such a probability distribution exists!
  - x is in the (original) MST polytope
  - Caratheodory's theorem: x is a convex combination of $m + 1$ vertices of MST polytope
  - By integrality of MST polytope: x is the "expectation" of a probability distribution over spanning trees.
- Consequence of Ellipsoid algorithm: can compute such a decomposition of x efficiently!

Outline



The Maximum Flow Problem

Given a directed graph $G = (V, E)$ with capacities u_e on edges e, a source node s, and a sink node t, find a maximum flow from s to t respecting the capacities.

$$
\begin{array}{lll}
\text{maximize} & \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e & \\
\text{subject to} & \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, & \text{for } v \in V \setminus \{s, t\}. \\
& x_e \le u_e, & \text{for } e \in E. \\
& x_e \ge 0, & \text{for } e \in E.
\end{array}
$$

Can be computed either by solving the LP, or by a combinatorial algorithm such as Ford Fulkerson.

Primal LP

$$\max \sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e$$

s.t.

$$\sum_{e\in\delta^-(v)} x_e = \sum_{e\in\delta^+(v)} x_e, \qquad \forall v \in V \setminus \{s,t\}$$

$$x_e \le u_e, \qquad \forall e \in E.$$

$$x_e \ge 0, \qquad \forall e \in E.$$

Dual LP (Simplified)

$$\min \sum_{e\in E} u_e z_e$$

s.t.

$$y_v - y_u \le z_e, \qquad \forall e = (u,v) \in E.$$

$$y_s = 0$$

$$y_t = 1$$

$$z_e \ge 0, \qquad \forall e \in E.$$

- Dual solution describes fraction z_e of each edge to fractionally cut

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- Dual solution describes fraction z_e of each edge to fractionally cut
- Dual constraints require that at least 1 edge is cut on every path from s to t.
  - $\sum_{(u,v) \in P} z_{uv} \geq \sum_{(u,v) \in P} y_v - y_u = y_t - y_s = 1$

Primal LP

$$\max \sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e$$

s.t.

$$\sum_{e\in\delta^-(v)} x_e = \sum_{e\in\delta^+(v)} x_e, \qquad \forall v \in V \setminus \{s,t\}$$

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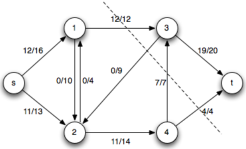

- Every integral $s-t$ cut is feasible.

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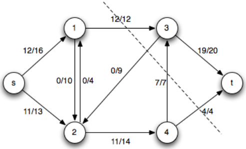

- Every integral $s-t$ cut is feasible.
- By weak duality: max flow $\le$ minimum cut

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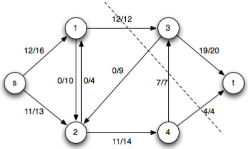

- Every integral $s-t$ cut is feasible.
- By weak duality: max flow $\le$ minimum cut
- Ford-Fulkerson shows that max flow $=$ min cut
  - i.e. dual has integer optimal

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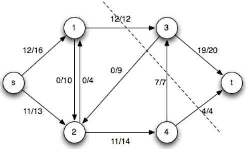

- Every integral $s-t$ cut is feasible.
- By weak duality: max flow $\leq$ minimum cut
- Ford-Fulkerson shows that max flow $=$ min cut
  - i.e. dual has integer optimal
- Ford-Fulkerson also shows that there is an integral optimal flow when capacities are integer.

Generalizations of Max Flow

$$\begin{aligned}
&\text{max} \sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e \\
&\text{s.t.} \\
&\sum_{e\in\delta^-(v)} x_e = \sum_{e\in\delta^+(v)} x_e, && \forall v \in V \setminus \{s,t\}. \\
&x_e \le u_e, && \forall e \in E. \\
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\end{aligned}$$

Writing as an LP shows that many generalizations are also tractable

Generalizations of Max Flow

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Writing as an LP shows that many generalizations are also tractable

- Lower and upper bound constraints on flow: $\ell_e \le x_e \le u_e$

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Writing as an LP shows that many generalizations are also tractable

- Lower and upper bound constraints on flow: $\ell_e \le x_e \le u_e$
- minimum cost flow of a certain amount r
  - Objective $\min \sum_e c_e x_e$
  - Additional constraint: $\sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e = r$

Generalizations of Max Flow

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- Multiple commodities sharing the network

Generalizations of Max Flow

$$
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- Multiple commodities sharing the network
- …

Minimum Congestion Flow

You are given a directed graph $G = (V, E)$ with congestion functions $c_e(.)$ on edges e, a source node s, a sink node t, and a desired flow amount r. Find a minimum average congestion flow from s to t.

$$
\begin{array}{lll}
\text{minimize} & \sum_e x_e c_e(x_e) & \\
\text{subject to} & \sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e = r & \\
 & \sum_{e\in\delta^-(v)} x_e = \sum_{e\in\delta^+(v)} x_e, & \text{for } v \in V \setminus \{s,t\}\,. \\
 & x_e \geq 0, & \text{for } e \in E.
\end{array}
$$

When $c_e(.)$ are polynomials with nonnegative co-efficients, e.g. $c_e(x) = a_e x^2 + b_e x + c_e$ with $a_e, b_e, c_e \geq 0$, this is a (non-linear) convex program.

Outline

The Max Cut Problem

Given an undirected graph $G = (V, E)$, find a partition of V into $(S, V \setminus S)$ maximizing number of edges with exactly one end in S.

$$\begin{array}{ll} \text{maximize} & \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2} \\ \text{subject to} & x_i \in \{-1, 1\}, \qquad \text{for } i \in V. \end{array}$$

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Instead of requiring x_i to be on the 1 dimensional sphere, we relax and permit it to be in the n-dimensional sphere, where $n = |V|$.

Vector Program relaxation

$$\begin{array}{lll} \text{maximize} & \sum_{(i,j)\in E} \frac{1-\vec{v}_i\cdot\vec{v}_j}{2} & \\ \text{subject to} & ||\vec{v}_i||_2 = 1, & \text{for } i \in V. \\ & \vec{v}_i \in \mathbb{R}^n, & \text{for } i \in V. \end{array}$$

SDP Relaxation

- Recall: A symmetric $n \times n$ matrix Y is PSD iff $Y = V^T V$ for $n \times n$ matrix V
- Equivalently: PSD matrices encode pairwise dot products of columns of V
- When diagonal entries of Y are 1, V has unit length columns
- Recall: Y and V can be recovered from each other efficiently

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Vector Program relaxation

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SDP Relaxation

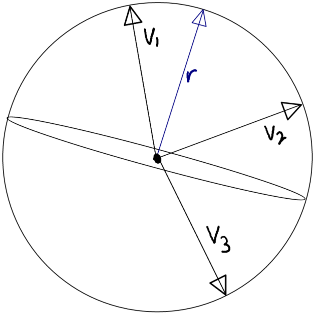
$$\begin{array}{lll} \text{maximize} & \sum_{(i,j)\in E} \frac{1-Y_{ij}}{2} & \\ \text{subject to} & Y_{ii} = 1, & \text{for } i \in V. \\ & Y \in S^n_+ & \end{array}$$

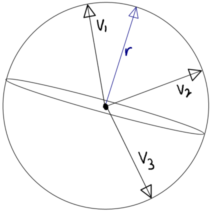
Goemans Williamson Algorithm for Max Cut

1. Solve the SDP to get $Y \succeq 0$
2. Decompose Y to VV^T
3. Draw random vector r on unit sphere
4. Place nodes i with $v_i \cdot r \geq 0$ on one side of cut, the rest on the other side

SDP Relaxation

$$\begin{array}{ll} \text{maximize} & \sum_{(i,j)\in E} \frac{1-Y_{ij}}{2} \\ \text{subject to} & Y_{ii} = 1 \; \forall i \\ & Y \in S^n_+ \end{array}$$

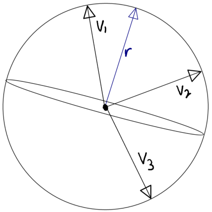




We will prove the following Lemma

Lemma

The random hyperplane cuts each edge (i, j) with probability at least $0.878\frac{1-Y_{ij}}{2}$



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Therefore, by linearity of expectations, and the fact that $OPT_{SDP} \geq OPT$ (i.e. relaxation).

Theorem

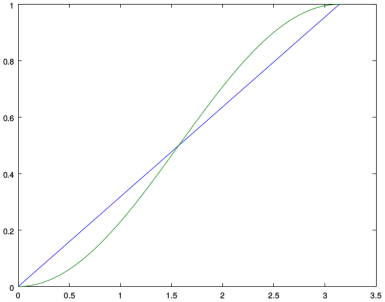
The Goemans Williamson algorithm outputs a random cut of expected size at least $0.878\,OPT$.

We use the following fact

Fact

For all angles $\theta \in [0, \pi]$,

$$\frac{\theta}{\pi} \geq 0.878 \cdot \frac{1 - cos(\theta)}{2}$$



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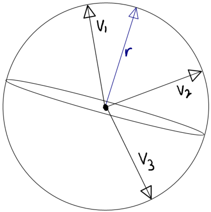

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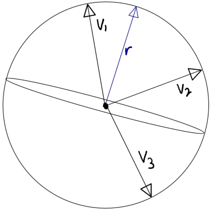

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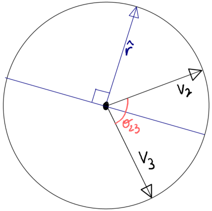

- (i, j) is cut iff $sign(r \cdot v_i) \neq sign(r \cdot v_j)$
- Can zoom in on the 2-d plane which includes v_i and v_j
  - Discard component r perpendicular to that plane, leaving $\widehat{r}$
  - Direction of $\widehat{r}$ is uniform in the plane

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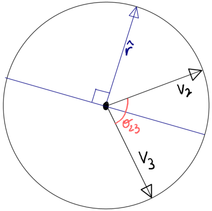
The random hyperplane cuts each edge (i, j) with probability at least $0.878\frac{1-Y_{ij}}{2}$



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- Let θ_{ij} be angle between v_i and v_j. Note $Y_{ij} = v_i \cdot v_j = \cos(\theta_{ij})$

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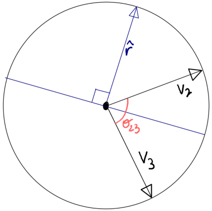

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- Can zoom in on the 2-d plane which includes v_i and v_j
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  - Direction of $\widehat{r}$ is uniform in the plane
- Let θ_{ij} be angle between v_i and v_j. Note $Y_{ij} = v_i \cdot v_j = \cos(\theta_{ij})$
- $\widehat{r}$ cuts (i, j) w.p.

$$\frac{2\theta_{ij}}{2\pi} = \frac{\theta_{ij}}{\pi} \geq 0.878\frac{1 - \cos\theta_{ij}}{2} = 0.878\frac{1 - Y_{ij}}{2}$$