CS675: Convex and Combinatorial Optimization
Spring 2022
Duality of Convex Optimization Problems

Instructor: Shaddin Dughmi

Outline

Recall: Optimization Problem in Standard Form

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & \text{for } i = 1, \ldots, m. \\ & h_i(x) = 0, & \text{for } i = 1, \ldots, k. \end{array}$$

- For convex optimization problems in standard form, f_i is convex and h_i is affine.
- Let $\mathcal{D}$ denote the domain of all these functions (i.e. when their value is finite)

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This Lecture + Next

We will develop duality theory for convex optimization problems, generalizing linear programming duality.

Running Example: Linear Programming

We have already seen the standard form LP below

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \preceq b \\ & x \succeq 0 \end{array}$$

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Along the way, we will recover the following standard form dual

$$\begin{array}{ll} \text{minimize} & y^\intercal b \\ \text{subject to} & A^\intercal y \succeq c \\ & y \succeq 0 \end{array}$$

The Lagrangian

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & \text{for } i = 1, \ldots, m. \\ & h_i(x) = 0, & \text{for } i = 1, \ldots, k. \end{array}$$

Basic idea of Lagrangian duality is to relax/soften the constraints by replacing each with a linear "penalty term" or "cost" in the objective.

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The Lagrangian Function

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} \nu_i h_i(x)$$

- λ_i is Lagrange Multiplier for i'th inequality constraint
  - Required to be nonnegative
- ν_i is Lagrange Multiplier for i'th equality constraint
  - Allowed to be of arbitrary sign

The Lagrange Dual Function

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad \text{for } i = 1, \ldots, m. \\ & h_i(x) = 0, \quad \text{for } i = 1, \ldots, k. \end{array}$$

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The Lagrange Dual Function

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} \nu_i h_i(x) \right)$$

- Observe: g is a concave function of the Lagrange multipliers
- We will see: Its quite common for the Lagrange dual to be unbounded $(-\infty)$ for some λ and ν
- By convention, domain of g is (λ, ν) s.t. $g(\lambda, \nu) > -\infty$

Langrange Dual of LP

$$\begin{array}{ll} \text{minimize} & -c^\intercal x \\ \text{subject to} & Ax - b \preceq 0 \\ & -x \preceq 0 \end{array}$$

First, the Lagrangian function

$$\begin{aligned} L(x, \lambda) &= -c^\intercal x + \lambda_1^\intercal (Ax - b) - \lambda_2^\intercal x \\ &= (A^\intercal \lambda_1 - c - \lambda_2)^\intercal x - \lambda_1^\intercal b \end{aligned}$$

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And the Lagrange Dual

$$\begin{aligned} g(\lambda) &= \inf_x L(x, \lambda) \\ &= \begin{cases} -\infty & \text{if } A^\intercal \lambda_1 - c - \lambda_2 \neq 0 \\ -\lambda_1^\intercal b & \text{if } A^\intercal \lambda_1 - c - \lambda_2 = 0 \end{cases} \end{aligned}$$

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So we restrict the domain of g to λ satisfying $A^\intercal \lambda_1 - c - \lambda_2 = 0$

Interpretation: "Soft" Lower Bound

$$\begin{array}{lll} \text{min} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & \text{for } i = 1, \ldots, m. \\ & h_i(x) = 0, & \text{for } i = 1, \ldots, k. \end{array}$$

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Fact

$g(\lambda, \nu)$ is a lowerbound on OPT(primal) for every $\lambda \succeq 0$ and $\nu \in \mathbb{R}^k$.

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Proof

- Every primal feasible x incurs nonpositive penalty by $L(x, \lambda, \nu)$
- Therefore, $L(x^*, \lambda, \nu) \leq f_0(x^*)$
- So $g(\lambda, \nu) \leq f_0(x^*) = OPT(Primal)$

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Interpretation

- A "hard" feasibility constraint can be thought of as imposing a penalty of $+\infty$ if violated, and a penalty/reward of 0 if satisfied
- Lagrangian imposes a "soft" linear penalty for violating a constraint, and a reward for slack
- Lagrange dual finds the optimal subject to these soft constraints

Interpretation: Geometric

Most easily visualized in the presence of a single inequality constraint

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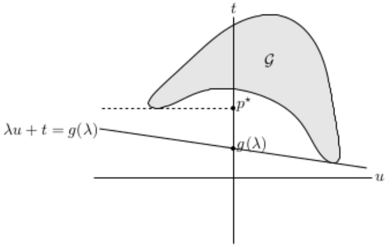

- Let $\mathcal{G}$ be attainable constraint/objective function value tuples
  - i.e. $(u, t) \in \mathcal{G}$ if there is an x such that $f_1(x) = u$ and $f_0(x) = t$
- $p^* = \inf \{t : (u, t) \in \mathcal{G}, u \le 0\}$
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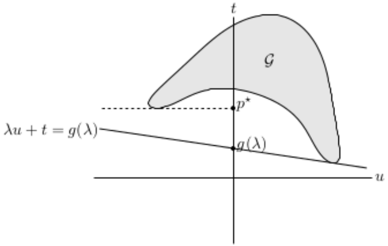

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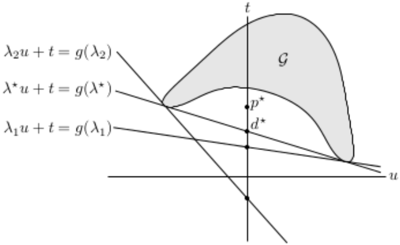
- $\lambda u + t = g(\lambda)$ is a supporting hyperplane to $\mathcal{G}$ pointing northeast
- Must intersect vertical axis below p^*
- Therefore $g(\lambda) \leq p^*$

The Lagrange Dual Problem

This is the problem of finding the best lower bound on OPT(primal) implied by the Lagrange dual function

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$



- Note: this is a convex optimization problem, regardless of whether primal problem was convex
- By convention, sometimes we add "dual feasibility" constraints to impose "nontrivial" lowerbounds (i.e. $g(\lambda, \nu) \geq -\infty$)
- (λ^*, ν^*) solving the above are referred to as the dual optimal solution

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Recall

Our Lagrange dual function for the above minimization LP (to the right), defined over the domain $A^\intercal \lambda_1 - c - \lambda_2 = 0$.

$$g(\lambda) = -\lambda_1^\intercal b$$

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$-$maximize $-\lambda_1^\intercal b$

subject to ~~$A^\intercal \lambda_1 - c - \lambda_2 = 0$~~

$A^\intercal \lambda_1 \succeq c$

$\lambda \succeq 0$

Another Example: Conic Optimization Problem

$$\begin{array}{ll} \text{minimize} & c^\intercal x \\ \text{subject to} & Ax = b \\ & x \in K \end{array}$$

- $x \in K$ can equivalently be written as $z^\intercal x \leq 0, \forall z \in K^\circ$

$$\begin{aligned} L(x, \lambda, \nu) &= c^\intercal x + \nu^\intercal (Ax - b) + \sum_{z \in K^\circ} \lambda_z \cdot z^\intercal x \\ &= (c - A^\intercal \nu + \sum_{z \in K^\circ} \lambda_z \cdot z)^\intercal x + \nu^\intercal b \end{aligned}$$

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- Can think of $\lambda \succeq 0$ as choosing some $s \in K^\circ$

$$L(x, s, \nu) = (c - A^\intercal \nu + s)^\intercal x + \nu^\intercal b$$

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Outline

Weak Duality

Primal Problem

min $f_0(x)$
s.t.
$f_i(x) \leq 0, \quad \forall i = 1, \ldots, m.$
$h_i(x) = 0, \quad \forall i = 1, \ldots, k.$

Dual Problem

max $g(\lambda, \nu)$
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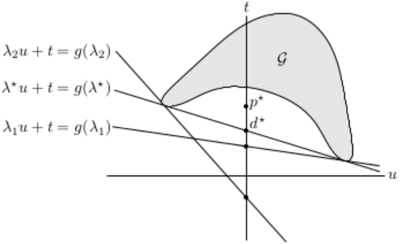
Weak Duality

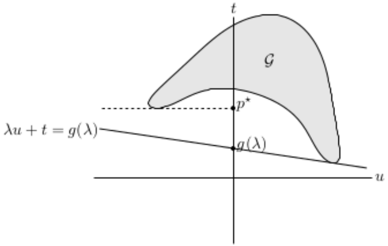
$OPT(dual) \leq OPT(primal)$.



- We have already argued holds for every optimization problem
- Duality Gap: difference between optimal dual and primal values

Recall: Geometric Interpretation of Weak Duality

minimize $f_0(x)$
subject to $f_1(x) \leq 0$



- Let $\mathcal{G}$ be attainable constraint/objective function value tuples
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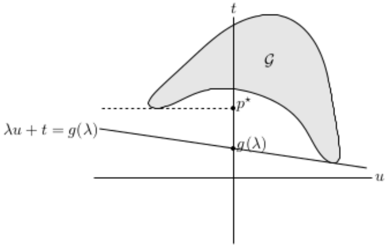

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Fact

The equation $\lambda u + t = g(\lambda)$ defines a supporting hyperplane to $\mathcal{G}$, intersecting t axis at $g(\lambda) \leq p^*$.

Strong Duality

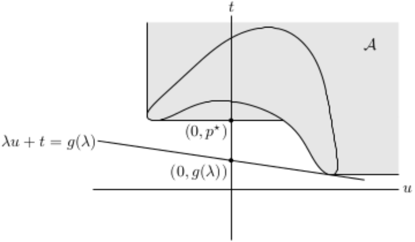
Strong Duality

We say strong duality holds if $OPT(dual) = OPT(primal)$.

- Equivalently: there exists a setting of Lagrange multipliers so that $g(\lambda, \nu)$ gives a tight lowerbound on primal optimal value.
- In general, does not hold for non-convex optimization problems
- Usually, but not always, holds for convex optimization problems.
  - Mild assumptions, such as Slater's condition, needed.

Geometric Proof of Strong Duality

minimize $\quad f_0(x)$
subject to $\quad f_1(x) \leq 0$



- Let $\mathcal{A}$ be everything northeast (i.e. "worse") than $\mathcal{G}$
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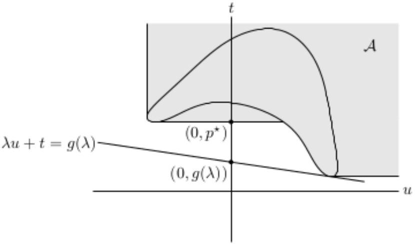

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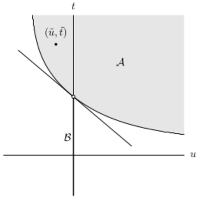

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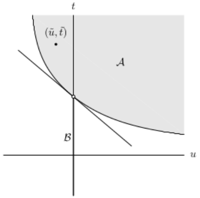

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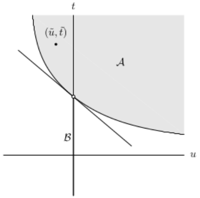
Proof

- Assume (u, t) and (u', t') are in $\mathcal{A}$

Geometric Proof of Strong Duality

minimize	$f_0(x)$
subject to	$f_1(x) \leq 0$



Fact

When f_0 and f_1 are convex, $\mathcal{A}$ is convex.

Proof

- Assume (u, t) and (u', t') are in $\mathcal{A}$
- $\exists x, x'$ with $(f_1(x), f_0(x)) \leq (u, t)$ and $(f_1(x'), f_0(x')) \leq (u', t')$.

Geometric Proof of Strong Duality

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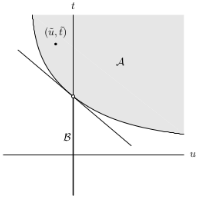

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  $(f_1(\alpha x + (1-\alpha)x'), f_0(\alpha x + (1-\alpha)x')) \leq (\alpha u + (1-\alpha)u', \alpha t + (1-\alpha)t')$

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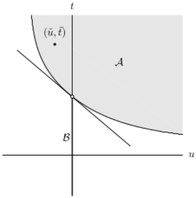

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- Therefore, segment connecting (u, t) and (u', t') also in $\mathcal{A}$.

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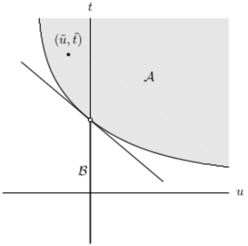

Theorem (Informal)

There is a choice of λ so that $g(\lambda) = p^*$. Therefore, strong duality holds.

Geometric Proof of Strong Duality

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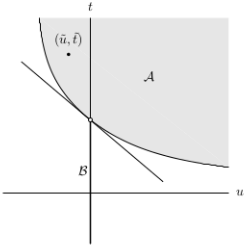


Theorem (Informal)

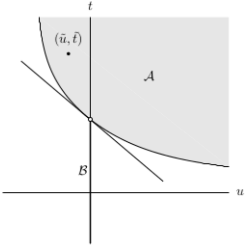
There is a choice of λ so that $g(\lambda) = p^*$. Therefore, strong duality holds.

Proof

- Recall $(0, p^*)$ is on the boundary of $\mathcal{A}$
- By the supporting hyperplane theorem, there is a supporting hyperplane to $\mathcal{A}$ at $(0, p^*)$
- Direction of the supporting hyperplane gives us an appropriate λ

I Lied (A little)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \leq 0 \end{array}$$



- In our proof, we ignored a technicality that can prevent strong duality from holding.

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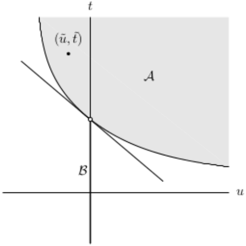

- In our proof, we ignored a technicality that can prevent strong duality from holding.
- What if our supporting hyperplane H at $(0, p^*)$ is vertical?
  - The normal to H is perpendicular to the t axis
- In this case, no finite λ exists such that $(\lambda, 1)$ is normal to H.

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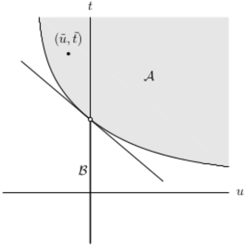

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- What if our supporting hyperplane H at $(0, p^*)$ is vertical?
  - The normal to H is perpendicular to the t axis
- In this case, no finite λ exists such that $(\lambda, 1)$ is normal to H.
- Somewhat counterintuitively, this can happen even in simple convex optimization problems (though its somewhat rare in practice)

Violation of Strong Duality

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & \frac{x^2}{y} \leq 0 \end{array}$$

- Let domain be the region $y \geq 1$
- Problem is convex, with feasible region given by $x = 0$
- Optimal value is 1, at $x = 0$ and $y = 1$

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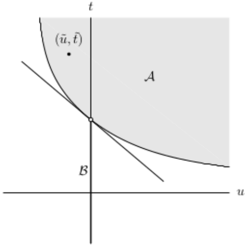
- Let domain be the region $y \geq 1$
- Problem is convex, with feasible region given by $x = 0$
- Optimal value is 1, at $x = 0$ and $y = 1$
- $\mathcal{A} = \mathbb{R}^2_{++} \bigcup (\{0\} \times [1, \infty])$
- Therefore, any supporting hyperplane to $\mathcal{A}$ at $(0, 1)$ must be vertical.
- Optimal dual value is 0; a duality gap of 1.

Slater's Condition

There exists a point $x \in \mathcal{D}$ where all inequality constraints are strictly satisfied (i.e. $f_i(x) < 0$). I.e. the optimization problem is strictly feasible.



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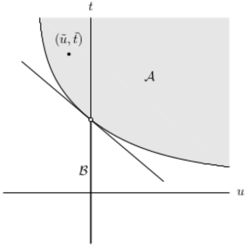

- A sufficient condition for strong duality.
- Forces supporting hyperplane to be non-vertical
- Can be weakened to requiring strict feasibility only of non-affine constraints

Outline

Recall: Lagrangian Duality

Primal Problem

min $f_0(x)$
s.t.
$f_i(x) \leq 0, \quad \forall i = 1, \ldots, m.$
$h_i(x) = 0, \quad \forall i = 1, \ldots, k.$

Dual Problem

max $g(\lambda, \nu)$
s.t.
$\lambda \succeq 0$

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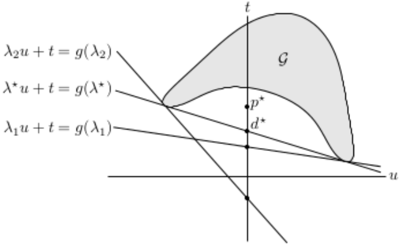

Weak Duality

$OPT(dual) \leq OPT(primal)$.

Recall: Lagrangian Duality

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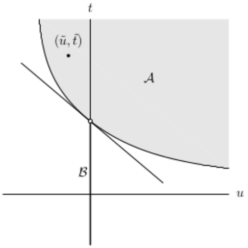
Strong Duality

$OPT(dual) = OPT(primal)$.



Dual Solution as a Certificate

Primal Problem

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- Dual solutions serves as a certificate of optimality
- If $f_0(x) = g(\lambda, \nu)$, and both are feasible, then both are optimal.

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- If $f_0(x) = g(\lambda, \nu)$, and both are feasible, then both are optimal.
- If $f_0(x) - g(\lambda, \nu) \leq \epsilon$, then both are within ϵ of optimality.
  - OPT(primal) and OPT(dual) lie in the interval $[g(\lambda, \nu), f_0(x)]$

Dual Solution as a Certificate

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  - OPT(primal) and OPT(dual) lie in the interval $[g(\lambda, \nu), f_0(x)]$
- Primal-dual algorithms use dual certificates to recognize optimality, or bound sub-optimality.

Implications of Strong Duality

Primal Problem

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Facts

If strong duality holds, and x^* and (λ^*, ν^*) are feasible & optimal, then

- x^* minimizes $L(x, \lambda^*, \nu^*)$ over all x.
- $\lambda_i^* f_i(x^*) = 0$ for all $i = 1, \ldots, m$. (Complementary Slackness)

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Proof

$$
\begin{aligned}
f_0(x^*) &= g(\lambda^*, \nu^*) = \min_x L(x, \lambda^*, \nu^*) \\
&\leq L(x^*, \lambda^*, \nu^*) = f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{k} \nu_i^* h_i(x^*) \\
&\leq f_0(x^*)
\end{aligned}
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Interpretation

- Lagrange multipliers (λ^*, ν^*) "simulate" the primal feasibility constraints
- Interpreting λ_i as the "value" of the i'th constraint, at optimality only the binding constraints are "valuable"
  - Recall economic interpretation of LP

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KKT Conditions

Suppose the primal problem is convex and defined on an open domain, and moreover the constraint functions are differentiable everywhere in the domain. If strong duality holds, then x^* and (λ^*, ν^*) are optimal iff:

- x^* and (λ^*, ν^*) are feasible
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Why are KKT Conditions Useful?

- Derive an analytical solution to some convex optimization problems
- Gain structural insights

Example: Equality-constrained Quadratic Program

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^\intercal P x + q^\intercal x + r \\ \text{subject to} & Ax = b \end{array}$$

- KKT Conditions: $Ax^* = b$ and $Px^* + q + A^\intercal \nu^* = 0$
- Simply a solution of a linear system with variables x^* and ν^*.
  - $m + n$ constraints and $m + n$ variables

Example: Market Equilibria (Fisher's Model)

- Buyers B, and goods G.
- Buyer i has utility u_{ij} for each unit of good G.
- Buyer i has budget m_i, and there's one divisible unit of each good.

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  - Prices p_j on items, such that each player can buy his favorite bundle that he can afford and the market clears (supply = demand).

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Eisenberg-Gale Convex Program

$$\begin{array}{lll} \text{maximize} & \sum_i m_i \log \sum_j u_{ij} x_{ij} & \\ \text{subject to} & \sum_i x_{ij} \leq 1, & \text{for } j \in G. \\ & x \succeq 0 & \end{array}$$

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Using KKT conditions, we can prove that the dual variables corresponding to the item supply constraints are market-clearing prices!