CS675: Convex and Combinatorial Optimization
Spring 2022
Convex Functions

Instructor: Shaddin Dughmi

Outline

Convex Functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if the line segment between any points on the graph of f lies above f. i.e. if $x, y \in \mathbb{R}^n$ and $\theta \in [0, 1]$, then

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

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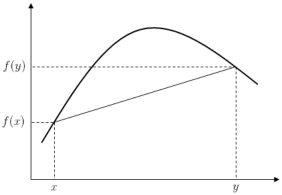
- f is strictly convex if inequality strict when $x \neq y$ and $\theta \in (0, 1)$.
- Analogous definition when domain of f is a convex subset of $\mathbb{R}^n$

Concave and Affine Functions



A function is $f : \mathbb{R}^n \to \mathbb{R}$ is concave if $-f$ is convex. Equivalently:

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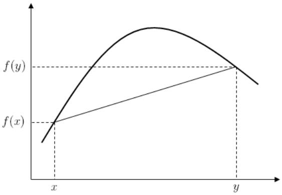

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$f : \mathbb{R}^n \to \mathbb{R}$ is affine if it is both concave and convex. Equivalently:

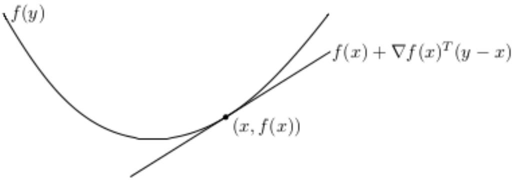
- Line segment between any points on the graph of f lies on the graph of f.
- $f(x) = a^{\intercal} x + b$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

We will now look at some equivalent definitions of convex functions

First Order Definition

A differentiable $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the first-order approximation centered at any point x underestimates f everywhere.

$$f(y) \geq f(x) + (\bigtriangledown f(x))^\intercal (y - x)$$



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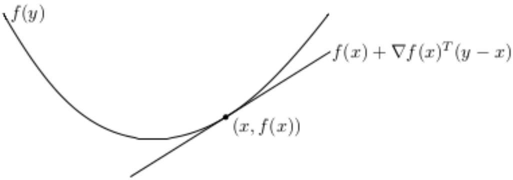

- Local information $\to$ global information
- If $\bigtriangledown f(x) = 0$ then x is a global minimizer of f

Second Order Definition

A twice differentiable $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its Hessian matrix $\bigtriangledown^2 f(x)$ is positive semi-definite for all x. (We write $\bigtriangledown^2 f(x) \succeq 0$)

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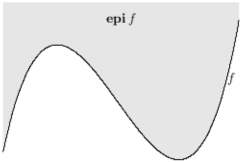
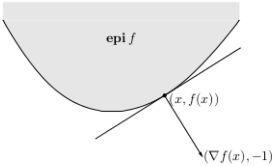
Intepretation

- Recall definition of PSD: $z^\intercal \bigtriangledown^2 f(x) z \geq 0$ for all $z \in \mathbb{R}^n$
- When $n = 1$, this is $f''(x) \geq 0$.
- More generally, $\frac{z^\intercal \bigtriangledown^2 f(x) z}{||z||^2}$ is the second derivative of f along the line $\{x + tz : t \in \mathbb{R}\}$. So if $\bigtriangledown^2 f(x) \succeq 0$ then f curves upwards along any line.
- Moving from x to $x + \delta\vec{z}$, infitisimal change in gradient is $\delta \bigtriangledown^2 f(x) z$. When $\bigtriangledown^2 f(x) \succeq 0$, this is in roughly the same direction as $\vec{z}$.

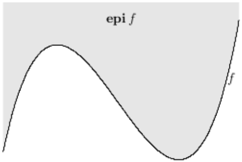


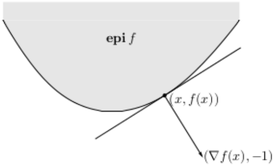


Epigraph

The epigraph of f is the set of points above the graph of f. Formally,

$$epi(f) = \{(x, t) : t \geq f(x)\}$$





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Epigraph Definition

f is a convex function if and only if its epigraph is a convex set.

Jensen's Inequality (General Form)

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if

- For every $x_1, \ldots, x_k$ in the domain of f, and $\theta_1, \ldots, \theta_k \geq 0$ such that $\sum_i \theta_i = 1$, we have

$$f(\sum_i \theta_i x_i) \leq \sum_i \theta_i f(x_i)$$

- Given a probability measure $\mathcal{D}$ on the domain of f, and $x \sim \mathcal{D}$,

$$f(\mathbf{E}[x]) \leq \mathbf{E}[f(x)]$$

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Adding noise to x can only increase $f(x)$ in expectation.

Local and Global Optimality

Local minimum

x is a local minimum of f if there is a an open ball B containing x where $f(y) \geq f(x)$ for all $y \in B$.

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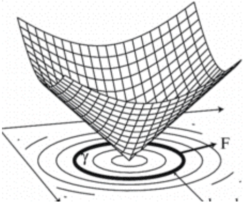
- This fact underlies much of the tractability of convex optimization.

Sub-level sets

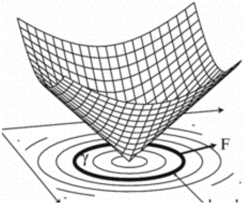


Level sets of $f(x, y) = \sqrt{x^2 + y^2}$

Sublevel set

The α-sublevel set of f is $\{x \in domain(f) : f(x) \leq \alpha\}$.

Sub-level sets



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Every sub-level set of a convex function is a convex set.

- This fact also underlies tractability of convex optimization

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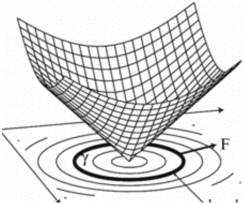

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Note: converse false, but nevertheless useful check.

Other Basic Properties

Continuity

Real-valued convex functions are continuous on the interior of their domain.

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Extended-value extension

If a function $f : D \to \mathbb{R}$ is convex on its domain, and D is convex, then it can be extended to a convex function on $\mathbb{R}^n$ by setting $f(x) = \infty$ whenever $x \notin D$.

This simplifies notation. Resulting function $\widetilde{f} : D \to \mathbb{R} \bigcup \infty$ is "convex" with respect to the ordering on $\mathbb{R} \bigcup \infty$

Outline

Functions on the reals

- Affine: $ax + b$
- Exponential: e^{ax} convex for any $a \in \mathbb{R}$
- Powers: x^a convex on $\mathbb{R}_{++}$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$
- Logarithm: $\log x$ concave on $\mathbb{R}_{++}$.

Norms

Norms are convex.

$$||\theta x + (1-\theta)y|| \leq ||\theta x|| + ||(1-\theta)y|| = \theta||x|| + (1-\theta)||y||$$

- Uses both norm axioms: triangle inequality, and homogeneity.
- Applies to matrix norms, such as the spectral norm (radius of induced ellipsoid)

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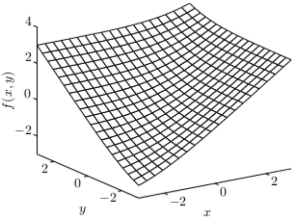
Max

$\max_i x_i$ is convex

$$\begin{aligned}\max_i(\theta x + (1-\theta)y)_i &= \max_i(\theta x_i + (1-\theta)y_i) \\ &\leq \max_i \theta x_i + \max_i (1-\theta)y_i \\ &= \theta \max_i x_i + (1-\theta)\max_i y_i\end{aligned}$$

If i'm allowed to pick the maximum entry of θx and θy independently, I can do only better.

- Log-sum-exp: $\log(e^{x_1} + e^{x_2} + \ldots + e^{x_n})$ is convex
- Geometric mean: $(\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$ is concave
- Log-determinant: $\log\det X$ is concave
- Quadratic form: $x^\intercal A x$ is convex iff $A \succeq 0$
- Other examples in book



$f(x, y) = \log(e^x + e^y)$

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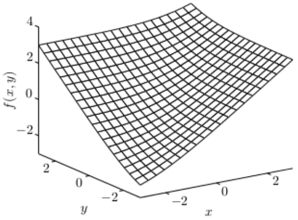

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Proving convexity often comes down to case-by-case reasoning, involving:

- Definition: restrict to line and check Jensen's inequality
- Write down the Hessian and prove PSD
- Express as a combination of other convex functions through convexity-preserving operations (Next)

Outline

Nonnegative Weighted Combinations

If $f_1, f_2, \ldots, f_k$ are convex, and $w_1, w_2, \ldots, w_k \geq 0$, then $g = w_1 f_1 + w_2 f_2 \ldots + w_k f_k$ is convex.

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proof $(k = 2)$

$$
\begin{aligned}
g\left(\frac{x+y}{2}\right) &= w_1 f_1\left(\frac{x+y}{2}\right) + w_2 f_2\left(\frac{x+y}{2}\right) \\
&\leq w_1 \frac{f_1(x) + f_1(y)}{2} + w_2 \frac{f_2(x) + f_2(y)}{2} \\
&= \frac{g(x) + g(y)}{2}
\end{aligned}
$$

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Extends to integrals $g(x) = \int_y w(y) f_y(x)$ with $w(y) \geq 0$, and therefore expectations $\mathbf{E}_y \, f_y(x)$.

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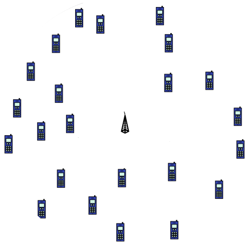
Worth Noting

Minimizing the expectation of a random convex cost function is also a convex optimization problem!

- A stochastic convex optimization problem is a convex optimization problem.

Example: Stochastic Facility Location



Average Distance

- k customers located at $y_1, y_2, \ldots, y_k \in \mathbb{R}^n$
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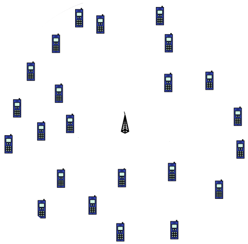

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- Since distance to any one customer is convex in x, so is the average distance.
- Extends to probability measure over customers

Implication

Convex functions are a convex cone in the vector space of functions from $\mathbb{R}^n$ to $\mathbb{R}$.

The set of convex functions is the intersection of an infinite set of homogeneous linear inequalities indexed by x, y, θ

$$f(\theta x + (1-\theta)y) - \theta f(x) - (1-\theta)f(y) \leq 0$$

Composition with Affine Function

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, and $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$, then

$$g(x) = f(Ax + b)$$

is a convex function from $\mathbb{R}^m$ to $\mathbb{R}$.

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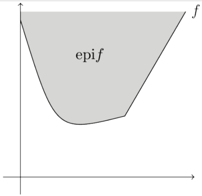

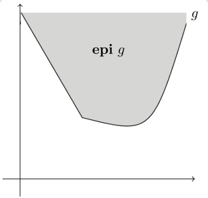

Proof

$$(x, t) \in \mathbf{graph}(g) \iff t = g(x) = f(Ax+b) \iff (Ax+b, t) \in \mathbf{graph}(f)$$

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$$(x, t) \in \mathbf{epi}(g) \iff t \geq g(x) = f(Ax + b) \iff (Ax + b, t) \in \mathbf{epi}(f)$$

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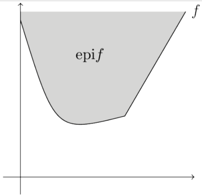

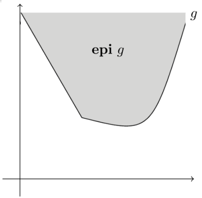

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$$(x,t) \in \mathbf{epi}(g) \iff t \geq g(x) = f(Ax+b) \iff (Ax+b,t) \in \mathbf{epi}(f)$$

$\mathbf{epi}(g)$ is the inverse image of $\mathbf{epi}(f)$ under the affine mapping $(x,t) \to (Ax+b,t)$

Examples

- $||Ax + b||$ is convex
- $\max(Ax + b)$ is convex
- $\log(e^{a_1^\intercal x + b_1} + e^{a_2^\intercal x + b_2} + \ldots + e^{a_n^\intercal x + b_n})$ is convex

Maximum

If f_1, f_2 are convex, then $g(x) = \max\{f_1(x), f_2(x)\}$ is also convex.

Generalizes to the maximum of any number of functions, $\max_{i=1}^{k} f_i(x)$, and also to the supremum of an infinite set of functions $\sup_y f_y(x)$.

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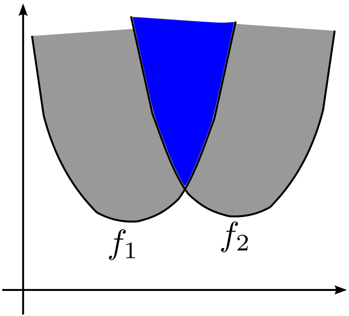
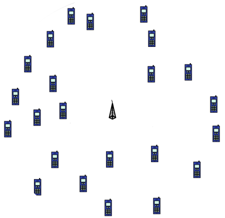


$$\mathbf{epi}\, g = \mathbf{epi}\, f_1 \bigcap \mathbf{epi}\, f_2$$

Example: Robust Facility Location



Maximum Distance

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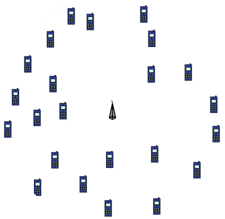

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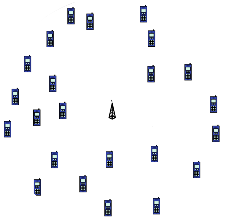

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Worth Noting

When a convex cost function is uncertain, minimizing the worst-case cost is also a convex optimization problem!

- A robust (in the worst-case sense) convex optimization problem is a convex optimization problem.

Other Examples

- Maximum eigenvalue of a symmetric matrix A is convex in A

$$\max\{v^\intercal Av : ||v|| = 1\}$$

- Sum of k largest components of a vector x is convex in x

$$\max\left\{\vec{\mathbf{1}}_S \cdot x : |S| = k\right\}$$

Minimization

If $f(x, y)$ is convex and $\mathcal{C}$ is convex and nonempty, then $g(x) = \inf_{y \in C} f(x, y)$ is convex.

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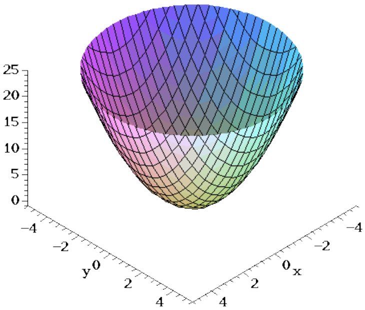
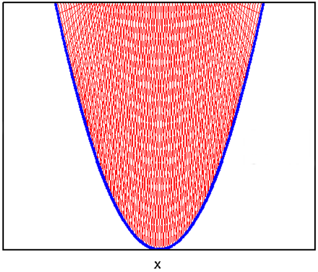
Proof (for $\mathcal{C} = \mathbb{R}^k$)

$\mathbf{epi}\, g$ is the projection of $\mathbf{epi}\, f$ onto hyperplane $y = 0$.



$$f(x, y) = x^2 + y^2$$



$$g(x) = x^2$$

Example

Distance from a convex set $\mathcal{C}$

$$f(x) = \inf_{y \in \mathcal{C}} ||x - y||$$

Composition Rules

If $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}$, then $f = h \circ g$ is convex if

- g_i are convex, and h is convex and nondecreasing in each argument.
- g_i are concave, and h is convex and nonincreasing in each argument.

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Proof ($n = k = 1$, twice differentiable)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

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Proof of first case

$$
\begin{aligned}
g(\theta x + (1-\theta)y) &\preceq \theta g(x) + (1-\theta)g(y) && \text{(component-wise)} \\
h(g(\theta x + (1-\theta)y)) &\leq h(\theta g(x) + (1-\theta)g(y)) && (h \text{ non-decreasing}) \\
&\leq \theta h(g(x)) + (1-\theta)h(g(y)) && (h \text{ convex})
\end{aligned}
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&\leq \theta h(g(x)) + (1-\theta) h(g(y)) && \text{(h convex)}
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Proof of second case is almost identical

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If f is convex then $g(x, t) = tf(x/t)$ is also convex.

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$\mathbf{epi}\, g$ is inverse image of $\mathbf{epi}\, f$ under the perspective function $(x,t,y) \to (x/t, y/t)$.