CSCI 699: Topics in Learning and Game Theory
Fall 2017
Lecture 3: Intro to Game Theory

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Outline

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Game Theory

The study of mathematical models of conflict and cooperation between rational decision-makers .

- Game: situation in which multiple decision makers (a.k.a. agents or players) make choices which influence each others' welfare (a.k.a. utility)
- Goal: Make predictions on how agents behave in a game

Game Theory

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- Game: situation in which multiple decision makers (a.k.a. agents or players) make choices which influence each others' welfare (a.k.a. utility)
- Goal: Make predictions on how agents behave in a game
- How to define rational decision-making: domain of decision theory
  - Special case of game theory for a single agent.

Bayesian Decision Theory

- There is a set Ω of future states of the world (e.g. times at which you might get to school)
- Set A of possible actions (e.g. which route you drive)
- For each $a \in A$, there is distribution $x(a)$ over outcomes Ω
  - Agent believes that he will receive $\omega \sim x(a)$ if he takes action a.

Question

How does a “rational” agent choose an action?

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Answer (Expected Utility Theory)

Agent has a subjective utility function $u : \Omega \to \mathbb{R}$, and chooses an action $a^* \in A$ maximizing his expected utility.

- $a^* \in \operatorname{argmax} \mathbf{E}_{\omega \sim x(a)}[u(\omega)]$
- If there are multiple such actions, may randomize among them arbitrarily.

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In addition to being simple/natural model, follows from VNM axioms.

Outline

Example: Rock, Paper, Scissors

	Rock	Paper	Scissors
Rock	0 0	+1 -1	-1 +1
Paper	-1 +1	0 0	+1 -1
Scissors	+1 -1	-1 +1	0 0

Rock, Paper, Scissors is an example of the most basic type of game.

Simultaneous move, complete information games

- Players act simultaneously
- Each player incurs a utility, determined by his action as well as the actions of others.
  - Players' actions determine "state of the world" or "outcome of the game".
- The payoff structure of the game, i.e. the map from action profiles to utility vectors, is common knowledge

Standard mathematical representation of such games:

Normal Form

An n-player game in normal form is given by

- A set of players $N = \{1, \ldots, n\}$.
- For each player i, a set of actions A_i.
  - Let $A = A_1 \times A_2 \times \ldots \times A_n$ denote the set of action profiles.
- For each player i, a utility function $u_i : A \to \mathbb{R}$.
  - If players play $a_1, \ldots, a_n$, then $u_i(a_1, \ldots, a_n)$ is the utility of player i.

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- Typically thought of as an n-dimensional matrix, indexed by $a = (a_1, \ldots, a_n) \in A$, with entry $(u_1(a), \ldots, u_n(a))$.
- Also useful for representing more general games, like sequential and incomplete information games, but is less natural there.

Strategies in Normal Form Games

- Pure strategy of player i: an action $a_i \in A_i$
  - Example: rock
- Mixed strategy of player i: a distribution s_i supported on A_i.
  - The player draws an action $a_i \sim s_i$.
  - Example: uniformly randomly choose one of rock, paper, scissors
- (pure/mixed) strategy profile $(s_1, \ldots, s_n)$: strategy for each player

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  - In most cases we discuss: independent. (Nash equilibrium)
  - Sometimes, they have a way of correlating (coordinating) their random choice (Correlated equilibrium)
- When we refer to mixed strategy profiles, we mean independent randomization unless stated otherwise.

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- When we refer to mixed strategy profiles, we mean independent randomization unless stated otherwise.
- We can extend utility functions to mixed strategy profiles. Using $s(a)$ as shorthand for the probability of action a in strategy s,

$$u_i(s_1, \ldots, s_n) = \sum_{a \in A} u_i(a) \prod_{j=1}^{n} s_j(a_j)$$

A mixed strategy s_i of player i is a best response to a strategy profile s_{-i} of the other players if $u_i(s) \geq u_i(s'_i, s_{-i})$ for every other mixed strategy s'_i.

- Note: There is always a pure best response
- The set of mixed best responses is the randomizations over pure best responses.

Example: Prisoner's Dilemma

	Coop	Defect
Coop	-1, -1	-3, 0
Defect	0, -3	-2, -2

Example: Battle of the Sexes

M \ W	Football	Movie
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Movie	0 0	2 1

Example: First Price Auction

Two players, with values $v_1 = 1$ and $v_2 = 2$, both common knowledge.

- $A_1 = A_2 = \mathbb{R}$ (note: infinite!)
- $u_i(a_1, a_2) = v_i - a_i$ if $a_i > a_{-i}$, and 0 otherwise.

Aside: Sequential Games

But . . .

what about “sequential” games which unfold over time, like chess, english auction, multiplayer video games, life, etc?

- More naturally modeled using the extensive form tree representation
  - Each non-leaf node is a step in the game, associated with a player
  - Outgoing edges = actions available at that step
  - leaf nodes labelled with utility of each player
  - Pure strategy: choice of action for each contingency (i.e. each non-leaf node)

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- Can be represented as a normal form game by collapsing pure strategies to actions of a large normal form game
  - Not as useful as extensive form.
- Won't need these games much in this class, though it's a good idea to know they exist, and maybe skim the chapter.

Equilibrium Concepts

An equilibrium concept identifies, for every game, one or more distributions over action profiles (the equilibria). Predicts that the outcome of the game is distributed as one of the equilibria.

For complete information games, here are the most notable in order of generality

- Dominant strategy equilibrium
- Nash equilibrium
- Correlated equilibrium
- Coarse-correlated equilibrium

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For sequential games, subgame-perfect equilibrium is the most pertinent. Sits between DSE and Nash.

Dominant-strategy Equilibrium

A strategy s_i of player i is a dominant strategy if it is a best response to every strategy profile s_{-i} of the other players. Formally, for all profiles s_{-i} of players other than i, we should have that $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for any other strategy s'_i of player i.

- If exists, i doesn't need to know what others are doing to respond.

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    - Mixed DS is randomization over pure DS

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A dominant-strategy equilibrium is a strategy profile where each player plays a dominant strategy.

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- Best kind of equilibrium (minimal knowledge assumptions)
- May be pure or mixed (independent randomization). Though mixed DSE are not of much interest.

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- Every DSE is also a Nash Equilibrium

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Dominant strategy: both players defect

Nash Equilibrium

A Nash equilibrium is a strategy profile $(s_1, \ldots, s_n)$ such that, for each player i, s_i is a best response to s_{-i}.

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- All players are optimally responding to each other, simultaneously.

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- If each s_i is pure we call it a pure Nash equilibrium, otherwise we call it a mixed Nash equilibrium.
- All players are optimally responding to each other, simultaneously.
- There might be a mixed Nash but no pure Nash (e.g. rock paper scissors)
- Every Nash equilibrium is a correlated equilibrium

Example: Battle of the Sexes

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- Mixed Nash: M goes to Football w.p. $2/3$ and Movie w.p. $1/3$, and W goes to Movie w.p. $2/3$ and Football w.p. $1/3$

Correlated Equilibrium

- Now we look at an equilibrium in which players choose correlated mixed strategies.
- Way to think about it: A mediator (e.g. traffic light) makes correlated action recommendations

A Correlated equilibrium is distribution x over action profiles such that, for each player i and action $a_i^* \in A_i$, we have

$$\mathop{\mathbf{E}}_{a \sim x}[u_i(a) | a_i = a_i^*] \geq \mathop{\mathbf{E}}_{a \sim x}[u_i(a_i', a_{-i}) | a_i = a_i^*]$$

for all $a_i' \in A_i$.

- In other words: If all players other than i follow their recommendations, then when i is recommended a_i^*, his posterior payoff is maximized by following his recommendation as well.

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Correlated Equilibrium: Uniformly randomize between (Football, Football) and (Movie,Movie)

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	STOP	GO
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GO	-2 1	-10 -10

- No DSE

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  - What else?

Existence of Equilibria

- Pure Nash equilibria and dominant strategy equilibria do not always exist (e.g. rock paper scissors)
- However, mixed Nash equilibrium always exists when there is a finite number of players and actions!

Theorem (Nash 1951)

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- Generalizes to some infinite games (continuous, compact)
- Implies existence for correlated equilibria as well.
- Proof is difficult. Much easier in the special case of zero-sum games (LP duality)

Arguments for/against Nash Equilibrium

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In favor:

- Always exists
- Requires no correlation device
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Against:

- There might be many Nash equilibria. How do players choose one collectively?
- Sometimes, no natural behavioral dynamics which converge to a Nash eq
- Sometimes hard to compute (PPAD complete in general)
  - “If your laptop can’t find it then neither can the market” – Kamal Jain

Outline

- In settings of complete information, Nash equilibria are a defensible prediction of the outcome of the game.
- In many settings, as in auctions, the payoff structure of the game itself is private to the players.
- How can a player possibly play his part of the Nash equilibrium if he's not sure what the game is, and therefore where the equilibrium is?
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Example: First price auction

$v_1 = 3$, v_2 is either 1 or 2, and a bid must be multiple of $\epsilon > 0$. The following is a Nash equilibrium: $b_1 = v_2 + \epsilon$ and $b_2 = v_2$.

- Player 1's equilibrium bid depends on player 2's private information!

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To explicitly model uncertainty, and devise credible solution concepts that take it into account, games of incomplete information were defined.

Modeling Uncertainty

Two main approaches are used to model uncertainty:

1. Prior-free:
   - A player doesn't have any beliefs about the private data of others (other than possible values it may take), and therefore about their strategies.
   - Only consider a strategy to be a "credible" prediction for a player if it is a best response in every possible situation.
2. Bayesian Common Prior:
   - Players' private data is drawn from a distribution, which is common knowledge
   - Player only knows his private data, but knows the distribution of others'
   - Bayes-Nash equilibrium generalizes Nash to take into account the distribution.

Though there are other approaches. . .

Prior-free Games

An n-player game of strict incomplete information is given by

- A set of players $N = \{1, \ldots, n\}$.
- For each player i, a set of actions A_i.
  - Let $A = A_1 \times A_2 \times \ldots \times A_n$ denote the set of action profiles.
- For each player i, a set of types T_i
  - Let $T = T_1 \times T_2 \times \ldots \times T_n$ denote the set of type profiles.
- For each player i, a utility function $u_i : T \times A \to \mathbb{R}$
  - We focus on independent private values: $u_i : T_i \times A \to \mathbb{R}$
  - $u_i(t_i, \vec{a})$ is utility of i when he has type t_i and players play $\vec{a}$

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When each player has just one type, what is this?

Example: Vickrey (Second-price) Auction

- $A_i = \mathbb{R}$ is the set of possible bids of player i.
- $T_i = \mathbb{R}$ is the set of possible values for the item.
- For $v_i \in T_i$ and $b \in A$, we have $u_i(v_i, b) = v_i - b_{-i}$ if $b_i > b_{-i}$, otherwise 0.

Strategies in Incomplete Information Games

- Strategies of player i
  - Pure strategy $s_i : T_i \to A_i$: a choice of action $a_i \in A_i$ for every type $t_i \in T_i$.
    - Example: Truthful bidding (i.e., bidding your value)
    - Another example: Bidding half your value
  - Mixed strategy: a distribution $s_i(t_i)$ over actions A_i for each type $t_i \in T_i$
    - Example: Bidding b_i uniform in $[0, v_i]$

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Note

- In a strategy, player decides how to act based only on his private info (his type), and NOT on others' private info nor their actions (neither of which he knows).
- A strategy profile $(s_1, \ldots, s_n)$ describes what would happen in each "state of the world" (i.e., type profile) $(t_1, \ldots, t_n)$, regardless of the relative frequency of various states.
  - In state $(t_1, \ldots, t_n)$, players play $(s_1(t_1), \ldots, s_n(t_n))$.

Dominant Strategy Equilibrium

DS and DSE generalize naturally to incomplete information games

$s_i : T_i \to \Delta(A_i)$ is a dominant strategy for player i if, for all $t_i \in T_i$ and $a_{-i} \in A_{-i}$ and $a'_i \in A_i$,

$$u_i(t_i, (s_i(t_i), a_{-i})) \geq u_i(t_i, (a'_i, a_{-i}))$$

- Equivalently: $s_i(t_i)$ is a best response to $a_{-i} \sim s_{-i}(t_{-i})$, for all t_i, t_{-i}, s_{-i}, and realization a_{-i}.
- In order to choose his best response, player i only needs to know his own type t_i, but not the types t_{-i} of other players, nor their strategies s_{-i}.
- As in complete info case, if there is DS, then there is a pure DS (i.e., $s_i : T_i \to A_i$)

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Essentially same properties, pros/cons, as in complete info

- Exists precisely when each player has a dominant strategy
- If there is DSE, then there is a pure DSE (i.e., $s_i : T_i \to A_i$ for each i)
- Best kind of equilibrium (minimal knowledge assumptions)
- May be pure or mixed. Though mixed DSE are not of much interest.
- Every DSE is also a Bayes-Nash Equilibrium

Vickrey (second-price) Auction

Consider a Vickrey Auction with incomplete information.

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Claim

The truth-telling strategy is dominant for each player.

Prove as an exercise

Bayesian Games

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  - Let $T = T_1 \times T_2 \times \ldots \times T_n$ denote the set of type profiles.
- For each player i, a utility function $u_i : T \times A \to \mathbb{R}$
  - We focus on independent private values: $u_i : T_i \times A \to \mathbb{R}$
  - $u_i(t_i, \vec{a})$ is utility of i when he has type t_i and players play $\vec{a}$
- Common prior: distribution $\mathcal{D}$ of T.

Bayesian Games

An n-player Bayesian game of Incomplete information is given by

- A set of players $N = \{1, \ldots, n\}$.
- For each player i, a set of actions A_i.
  - Let $A = A_1 \times A_2 \times \ldots \times A_n$ denote the set of action profiles.
- For each player i, a set of types T_i
  - Let $T = T_1 \times T_2 \times \ldots \times T_n$ denote the set of type profiles.
- For each player i, a utility function $u_i : T \times A \to \mathbb{R}$
  - We focus on independent private values: $u_i : T_i \times A \to \mathbb{R}$
  - $u_i(t_i, \vec{a})$ is utility of i when he has type t_i and players play $\vec{a}$
- Common prior: distribution $\mathcal{D}$ of T.

Example: First Price Auction

- $A_i = T_i = [0, 1]$
- $\mathcal{D}$ draws each $v_i \in T_i$ uniformly and independently from $[0, 1]$.
- $u_i(v_i, b) = v_i - b_i$ if $b_i > b_{-i}$, or 0 if $b_i < b_{-i}$ (ignoring ties for now).

Bayes-Nash Equilibrium

As before, a pure strategy s_i for player i is a map from T_i to A_i. Now, we define the extension of pure Nash equilibrium to this setting.

A pure Bayes-Nash Equilibrium of a Bayesian Game of incomplete information is a set of strategies $s_1, \ldots, s_n$, where $s_i : T_i \to A_i$, such that for all i, $t_i \in T_i$, $a'_i \in A_i$ we have

$$\mathop{\mathbf{E}}_{t_{-i} \sim \mathcal{D} | t_i} u_i(t_i, s(t)) \geq \mathop{\mathbf{E}}_{t_{-i} \sim \mathcal{D} | t_i} u_i(t_i, (a'_i, s_{-i}(t_{-i})))$$

where the expectation is over t_{-i} drawn from $\mathcal{D}$ after conditioning on t_i.

- Mixed BNE defined analogously: allow $s_i : T_i \to \Delta(A_i)$, and take expectations accordingly in the definition.
- Note: Every dominant strategy equilibrium is also a Bayes-Nash Equilibrium
- But, unlike DSE, (mixed) BNE is guaranteed to exist.

Example: First Price Auction

Example: First Price Auction

- Two players
- $A_i = T_i = [0, 1]$
- $\mathcal{D}$ draws each $v_i \in T_i$ uniformly and independently from $[0, 1]$.
- $u_i(v_i, b) = v_i - b_i$ if $b_i > b_{-i}$ (winner), $= \frac{1}{2}(v_i - b_i)$ if $b_1 = b_2$ (tied), and $= 0$ otherwise (loser).

Exercise: Show that the strategies $b_i(v_i) = v_i/2$ form a pure Bayes-Nash equilibrium.

Existence of Bayes-Nash Equilibrium

Theorem

Every finite Bayesian game of incomplete information admits a mixed Bayes-Nash equilibrium.

- Can prove it using Nash's theorem.
- It so happens that, in most natural Bayesian games we look at, there will be a pure BNE.

Bayes Correlated Equilibrium

- There is also a natural generalization of Correlated equilibrium to Bayesian games, called the Bayes correlated equilibrium
- Every BNE is a BCE, therefore BCE always guaranteed to exist for finite Bayesian games.
- There are also interesting BCE which are not BNE
- Conceptually, gets harder to wrap your head around the BCE definition. But, we won't need it in this class (I think) so won't get into it.