CSCI 699: Topics in Learning and Game Theory
Fall 2017
Lecture 4: Mechanism Design Preliminaries

Instructor: Shaddin Dughmi

Outline

Outline

Single-item Allocation

- n players
- Player i's private data (type): $v_i \in \mathbb{R}_+$
- Outcome: choice of a winning player, and payment from each player
- Utility of a player for an outcome is his value for the outcome if he wins, less payment

Objectives: Revenue, welfare.

Single-item Allocation

First Price Auction

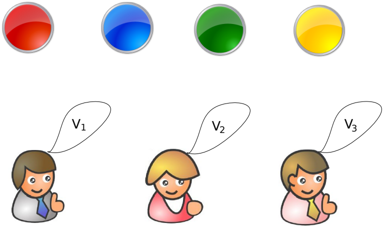
1. Collect bids
2. Give to highest bidder
3. Charge him his bid

Single-item Allocation

Second-price (Vickrey) Auction

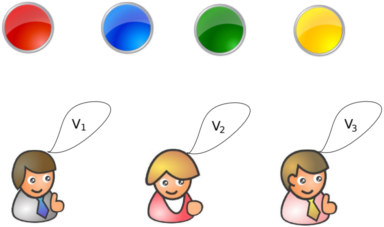
1. Collect bids
2. Give to highest bidder
3. Charge second highest bid

Example: Combinatorial Allocation



- n players, m items.
- Private valuation v_i : set of items $\rightarrow \mathbb{R}$.
  - $v_i(S)$ is player i's value for bundle S.

Example: Combinatorial Allocation

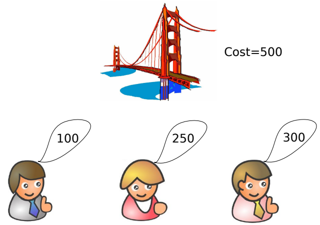


- n players, m items.
- Private valuation v_i : set of items $\to \mathbb{R}$.
  - $v_i(S)$ is player i's value for bundle S.

Goal

Partition items into sets $S_1, S_2, \ldots, S_n$ to maximize welfare:
$v_1(S_1) + v_2(S_2) + \ldots v_n(S_n)$

Example: Public Project



- n players
- Player i's private data (type): $v_i \in \mathbb{R}_+$
- Outcome: choice of whether or not to build, and payment from each player covering the cost of the project if built
- Utility of a player for an outcome is his value for the project if built, less his payment

Goal: Build if sum of values exceeds cost (maximize welfare), or maximize revenue

Example: Voting

- n players
- m candidates
- Player i's private data (type): total preference order on candidates
- Outcome: choice of winning candidate

Goal: ??

Outline

Mechanism Design Setting (Prior free)

Given by a tuple $(N, \mathcal{X}, T, u)$, where

- N is a finite set of players. Denote $n = |N|$ and $N = \{1, \ldots, n\}$.
- $\mathcal{X}$ is a set of outcomes.
- $T = T_1 \times \ldots T_n$, where T_i is the set of types of player i. Each $\vec{t} = (t_1, \ldots, t_n) \in T$ is called an type profile.
- $u = (u_1, \ldots u_n)$, where $u_i : T_i \times \mathcal{X} \to \mathbb{R}$ is the utility function of player i.

Mechanism Design Setting (Prior free)

Given by a tuple $(N, \mathcal{X}, T, u)$, where

- N is a finite set of players. Denote $n = |N|$ and $N = \{1, \ldots, n\}$.
- $\mathcal{X}$ is a set of outcomes.
- $T = T_1 \times \ldots T_n$, where T_i is the set of types of player i. Each $\vec{t} = (t_1, \ldots, t_n) \in T$ is called an type profile.
- $u = (u_1, \ldots u_n)$, where $u_i : T_i \times \mathcal{X} \to \mathbb{R}$ is the utility function of player i.

In a Bayesian setting, supplement with common prior $\mathcal{D}$ over T

Mechanism Design Setting (Prior free)

Given by a tuple $(N, \mathcal{X}, T, u)$, where

- N is a finite set of players. Denote $n = |N|$ and $N = \{1, \ldots, n\}$.
- $\mathcal{X}$ is a set of outcomes.
- $T = T_1 \times \ldots T_n$, where T_i is the set of types of player i. Each $\vec{t} = (t_1, \ldots, t_n) \in T$ is called an type profile.
- $u = (u_1, \ldots u_n)$, where $u_i : T_i \times \mathcal{X} \to \mathbb{R}$ is the utility function of player i.

In a Bayesian setting, supplement with common prior $\mathcal{D}$ over T

Example: Single-item Allocation

- Outcome: choice $x \in \{e_1, \ldots, e_n\}$ of winning player, and payment $p_1, \ldots, p_n$ from each
- Type of player i: value $v_i \in \mathbb{R}_+$.
- $u_i(v_i, x) = v_i x_i - p_i$.

Social Choice Functions

A principal wants to communicate with players and aggregate their private data (types) into a choice of outcome. Such aggregation captured by

A social choice function $f : T \to \mathcal{X}$ is a map from type profiles to outcomes.

Social Choice Functions

A principal wants to communicate with players and aggregate their private data (types) into a choice of outcome. Such aggregation captured by

A social choice function $f : T \to \mathcal{X}$ is a map from type profiles to outcomes.

Choosing a Social Choice Function

- A particular social choice function in mind (e.g. majority voting, utilitarian allocation of a single item, etc).
- An objective function $o : T \times \mathcal{X} \to \mathbb{R}$, and want $f(T)$ to (approximately) maximize $o(T, f(T))$
  - Either worst case over T (Prior-free) or in expectation (Bayesian)

Example: Single-item Allocation

- Welfare objective: $welfare(v, (x, p)) = \sum_i v_i x_i$
- Revenue objective: $revenue(v, (x, p)) = \sum_i p_i$

Mechanisms

To perform aggregation, principal runs protocol called a mechanism.

A mechanism is a pair (A, g), where

- $A = A_1 \times \ldots A_n$, where A_i is the set of possible actions (think messages, or bids) of player i in the protocol. $\mathcal{A}$ is the set of action profiles.
- $g : A \to \mathcal{X}$ is an outcome function

Mechanisms

To perform aggregation, principal runs protocol called a mechanism.

A mechanism is a pair (A,g), where

- $A = A_1 \times \ldots A_n$, where A_i is the set of possible actions (think messages, or bids) of player i in the protocol. $\mathcal{A}$ is the set of action profiles.
- $g : A \rightarrow \mathcal{X}$ is an outcome function

The resulting game of mechanism design is a game of incomplete information where when players play $a \in A$, player i's utility is $u_i(t_i, g(a))$ when his type is t_i.

Example: First price auction

- $A_i = \mathbb{R}$
- $g(b_1, \ldots, b_n) = (x, p)$ where $x_{i^*} = 1$, $p_{i^*} = b_{i^*}$ for $i^* = \text{argmax}_i\, b_i$, and $x_i = p_i = 0$ for $i \neq i^*$.

Implementation of Social Choice Functions

We say a mechanism (A, g) implements social choice function $f : T \to \mathcal{X}$ in dominant-strategy/Bayes-Nash equilibrium if there is a strategy profile $s = (s_1, \ldots, s_n)$ with $s_i : T_i \to A_i$ such that

- $s_i : T_i \to A_i$ is a dominant-strategy/Bayes-Nash equilibrium in the resulting incomplete information game
- $g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) = f(t_1, t_2, ..., t_n)$ for all $t \in T$

Example: First price, two players, i.i.d $U[0, 1]$

Implements in BNE the following social choice function: give the item to the player with the highest value and charges him half his value.

Example: Vickrey Auction

Implements in DSE the following social choice function: give the item to the player with the highest value and charges him the second highest value.

The Task of Mechanism Design

Task of Mechanism Design (Take 1)

Given a notion of a “good” social choice function from T to X, find

- A mechanism
  - An action space $A = (A_1, \ldots, A_n)$,
  - an outcome function $g : A \to \mathcal{X}$,
- an equilibrium $(s_1, \ldots, s_n)$ of the resulting game of mechanism design

such that the social choice function $f(t_1, \ldots, t_n) = g(s_1(t_1), \ldots, s_n(t_n))$ is “good.”

The Task of Mechanism Design

Task of Mechanism Design (Take 1)

Given a notion of a "good" social choice function from T to X, find

- A mechanism
  - An action space $A = (A_1, \ldots, A_n)$,
  - an outcome function $g : A \to \mathcal{X}$,
- an equilibrium $(s_1, \ldots, s_n)$ of the resulting game of mechanism design

such that the social choice function $f(t_1, \ldots, t_n) = g(s_1(t_1), \ldots, s_n(t_n))$ is "good."

Problem

This seems like a complicated, multivariate search problem.

The Task of Mechanism Design

Task of Mechanism Design (Take 1)

Given a notion of a "good" social choice function from T to X, find

- A mechanism
  - An action space $A = (A_1, \ldots, A_n)$,
  - an outcome function $g : A \to \mathcal{X}$,
- an equilibrium $(s_1, \ldots, s_n)$ of the resulting game of mechanism design

such that the social choice function $f(t_1, \ldots, t_n) = g(s_1(t_1), \ldots, s_n(t_n))$ is "good."

Problem

This seems like a complicated, multivariate search problem.

Luckily

The revelation principle reduces the search space to just $g : T \to \mathcal{X}$.

Outline

Incentive-Compatibility

Direct Revelation

A mechanism (A, g) is a direct revelation mechanism if $A_i = T_i$ for all i.

i.e. in a direct revelation mechanism, players simultaneously report types (not necessarily truthfully) to the mechanism. Such mechanisms can simply be described via the function $g : T \rightarrow \mathcal{X}$.

Incentive-Compatibility

A direct-revelation mechanism is dominant-strategy/Bayesian incentive-compatible (aka truthful) if the truth-telling is a dominant-strategy/Bayes-Nash equilibrium in the resulting incomplete-information game.

Note: A direct revelation incentive-compatible mechanism implements its outcome function $g : T \rightarrow \mathcal{X}$, by definition.

The social choice function IS the mechanism!!

Examples

Vickrey Auction

Direct revelation mechanism, dominant-strategy incentive-compatible.

First Price Auction

Direct revelation mechanism, not Bayesian incentive compatible.

Example: Posted price

The auction that simply posts a fixed price to players in sequence until one accepts is not direct revelation.

Revelation Principle

Revelation Principle

If there is a mechanism implementing social choice function f in dominant-strategy/Bayes-Nash equilibrium, then there is a direct revelation, dominant-strategy/Bayesian incentive-compatible mechanism implementing f.

Revelation Principle

Revelation Principle

If there is a mechanism implementing social choice function f in dominant-strategy/Bayes-Nash equilibrium, then there is a direct revelation, dominant-strategy/Bayesian incentive-compatible mechanism implementing f.

This simplifies the task of mechanism design

Task of Mechanism Design (Take 2)

Given a notion of a "good" social choice function from T to X, find such a function $f : T \to X$ such that truth-telling is an equilibrium in the following mechanism:

- Solicit reports $\widetilde{t_i} \in T_i$ from each player i (simultaneous, sealed bid)
- Choose outcome $f(\widetilde{t_1}, \ldots, \widetilde{t_n})$

Example

2 players, with values i.i.d uniform from $[0, 1]$, facing the first-price auction.

First-price Auction

1. Solicit bids b_1, b_2
2. Give item to highest bidder, charging him his bid

Recall

The strategies where each player reports half their value are in BNE. In other words, when player 1 knows his value v_1, and faces player 2 who is bidding uniformly from $[0, 1/2]$, he maximizes his expected utility $(v_1 - b_1).2b_1$ by bidding $b_1 = v_1/2$. And vice versa.

Example

2 players, with values i.i.d uniform from $[0, 1]$, facing the first-price auction.

First-price Auction

1. Solicit bids b_1, b_2
2. Give item to highest bidder, charging him his bid

Recall

The strategies where each player reports half their value are in BNE. In other words, when player 1 knows his value v_1, and faces player 2 who is bidding uniformly from $[0, 1/2]$, he maximizes his expected utility $(v_1 - b_1).2b_1$ by bidding $b_1 = v_1/2$. And vice versa.

Therefore ...

the first price auction implements in BNE the social choice function which gives the item to the highest bidder, and charges him half his bid

Example

Modified First-price Auction

1. Solicit bids b_1, b_2
2. Give item to highest bidder, charging him half his bid
   - Equivalently, simulate a first price auction where bidders bid $b_1/2, b_2/2$

Claim

Truth-telling is a BNE in the modified first-price auction.

Therefore, the modified auction implements the same social-choice function in equilibrium, but is truthful.

Example

Modified First-price Auction

1. Solicit bids b_1, b_2
2. Give item to highest bidder, charging him half his bid
   - Equivalently, simulate a first price auction where bidders bid $b_1/2, b_2/2$

Claim

Truth-telling is a BNE in the modified first-price auction.

Therefore, the modified auction implements the same social-choice function in equilibrium, but is truthful.

Proof

Assume player 2 bids truthfully. Player 1 faces a (simulated) first price auction where his own bid is halved before participating, and player 2 bids uniformly from $[0, 1/2]$. To respond optimally in the simulation, he bids $b_1 = v_1$ and lets the mechanism halve his bid on his behalf.

Proof (Bayesian Setting)

Consider mechanism (A, g), with BNE strategies $s_i : T_i \to A_i$.

- Implements $f(t_1, \ldots, t_n) = g(s_1(t_1), \ldots, s_n(t_n))$ in BNE
- For all i and t_i, action $s_i(t_i)$ maximizes player i's expected utility when other players are playing $s_{-i}(t_{-i})$ for $t_{-i} \sim \mathcal{D}|t_i$.

Proof (Bayesian Setting)

Consider mechanism (A, g), with BNE strategies $s_i : T_i \to A_i$.

- Implements $f(t_1, \ldots, t_n) = g(s_1(t_1), \ldots, s_n(t_n))$ in BNE
- For all i and t_i, action $s_i(t_i)$ maximizes player i's expected utility when other players are playing $s_{-i}(t_{-i})$ for $t_{-i} \sim \mathcal{D}|t_i$.

Modified Mechanism

1. Solicit reported types $\widetilde{t}_1, \ldots, \widetilde{t}_n$
2. Choose outcome $f(\widetilde{t}_1, \ldots, \widetilde{t}_n) = g(s_1(\widetilde{t}_1), \ldots, s_n(\widetilde{t}_n))$
   - Equivalently, simulate (A, g) when players play $s_i(t_i)$

Proof (Bayesian Setting)

Consider mechanism (A, g), with BNE strategies $s_i : T_i \to A_i$.

- Implements $f(t_1, \ldots, t_n) = g(s_1(t_1), \ldots, s_n(t_n))$ in BNE
- For all i and t_i, action $s_i(t_i)$ maximizes player i's expected utility when other players are playing $s_{-i}(t_{-i})$ for $t_{-i} \sim \mathcal{D}|t_i$.

Modified Mechanism

1. Solicit reported types $\widetilde{t}_1, \ldots, \widetilde{t}_n$
2. Choose outcome $f(\widetilde{t}_1, \ldots, \widetilde{t}_n) = g(s_1(\widetilde{t}_1), \ldots, s_n(\widetilde{t}_n))$
   - Equivalently, simulate (A, g) when players play $s_i(t_i)$

- Assume all players other than i report truthfully
- When i's type is t_i, other players playing $s_{-i}(t_{-i})$ for $t_{-i} \sim \mathcal{D}|t_i$ in simulated mechanism
- As stated above, his best response in simulation is $s_i(t_i)$.
- Mechanism transforms his bid by applying s_i, so best to bid t_i.

Outline

Incorporating Payments

To make much of modern mechanism design possible, we assume that

- The set of outcomes has a particular structure: every outcome includes a payment to or from each player.
- Player utilities vary linearly with their payment.

Examples: Single-item allocation, public project,
Non-examples: Single-item allocation without money, voting.

Quasilinear Utilities

The Quasi-linear Setting

Formally, $\mathcal{X} = \Omega \times \mathbb{R}^n$.

- Ω is the set of allocations
- For $(\omega, p_1, \ldots, p_n) \in \mathcal{X}$, p_i is the payment from (or to) player i.

and player $i's$ utility function $u_i : T_i \times \mathcal{X} \to \mathbb{R}$ takes the following form

$$u_i(t_i, (\omega, p_1, \ldots, p_n)) = v_i(t_i, \omega) - p_i$$

for some valuation function $v_i : T_i \times \Omega \to \mathbb{R}$.

We say players have quasilinear utilities.

Example: Single-item Allocation

- $\Omega = \{e_1, \ldots, e_n\}$
- $u_i(t_i, (\omega, p_1, \ldots, p_n)) = t_i \omega_i - p_i$

Further simplification

Recall that, using the revelation principle, we got

Task of Mechanism Design (Take 2)

Given a notion of a "good" social choice function from T to X, find such a function $f : T \to X$ such that truth-telling is an equilibrium in the following mechanism:

- Solicit reports $\widetilde{t_i} \in T_i$ from each player i (simultaneous, sealed bid)
- Choose outcome $f(\widetilde{t_1}, \ldots, \widetilde{t_n})$

Further simplification

In quasilinear settings this breaks down further

Task of Mechanism Design in Quasilinear settings

Find a "good" allocation rule $f : T \to \Omega$ and payment rule $p : T \to \mathbb{R}^n$ such that the following mechanism is incentive-compatible:

- Solicit reports $\widetilde{t_i} \in T_i$ from each player i (simultaneous, sealed bid)
- Choose allocation $f(\widetilde{t})$
- Charge player i payment $p_i(\widetilde{t})$

We think of the mechanism as the pair (f, p).
Sometimes, we abuse notation and think of type t_i directly as the valuation $v_i : \Omega \to \mathbb{R}$.

Incentive-Compatibility

Incentive compatibility can be stated simply now

Incentive-compatibility (Dominant Strategy)

A mechanism (f, p) is dominant-strategy truthful if, for every player i, true type t_i, possible mis-report $\widetilde{t}_i$, and reported types t_{-i} of the others, we have

$$v_i(t_i, f(t)) - p_i(t) \geq v_i(t_i, f(\widetilde{t}_i, t_{-i})) - p_i(\widetilde{t}_i, t_{-i})$$

If (f, p) randomized, add expectation signs.

Incentive-Compatibility

Incentive compatibility can be stated simply now

Incentive-compatibility (Dominant Strategy)

A mechanism (f, p) is dominant-strategy truthful if, for every player i, true type t_i, possible mis-report $\widetilde{t}_i$, and reported types t_{-i} of the others, we have

$$v_i(t_i, f(t)) - p_i(t) \geq v_i(t_i, f(\widetilde{t}_i, t_{-i})) - p_i(\widetilde{t}_i, t_{-i})$$

If (f, p) randomized, add expectation signs.

Incentive-compatibility (Bayesian)

A mechanism (f, p) is Bayesian incentive compatible if, for every player i, true type t_i, possible mis-report $\widetilde{t}_i$, the following holds in expectation over $t_{-i} \sim D|t_i$

$$\mathbf{E}[v_i(t_i, f(t)) - p_i(t)] \geq \mathbf{E}[v_i(t_i, f(\widetilde{t}_i, t_{-i})) - p_i(\widetilde{t}_i, t_{-i})]$$

Examples

Vickrey Auction

- Allocation rule maps $b_1, \ldots, b_n$ to e_{i^*} for $i^* = \mathrm{argmax}_i\, b_i$
- Payment rule maps $b_1, \ldots, b_n$ to $p_1, \ldots, p_n$ where $p_{i^*} = b_{(2)}$, and $p_i = 0$ for $i \neq i^*$.

Dominant-strategy truthful.

First Price Auction

- Allocation rule maps $b_1, \ldots, b_n$ to e_{i^*} for $i^* = \mathrm{argmax}_i\, b_i$
- Payment rule maps $b_1, \ldots, b_n$ to $p_1, \ldots, p_n$ where $p_{i^*} = b_{(1)}$, and $p_i = 0$ for $i \neq i^*$.

For two players i.i.d $U[0, 1]$, players bidding half their value is a BNE.
Not Bayesian incentive compatible.

Examples

Modified First Price Auction

- Allocation rule maps $b_1, \ldots, b_n$ to e_{i^*} for $i^* = \text{argmax}_i\, b_i$
- Payment rule maps $b_1, \ldots, b_n$ to $p_1, \ldots, p_n$ where $p_{i^*} = b_{(1)}/2$, and $p_i = 0$ for $i \neq i^*$.

For two players i.i.d $U[0, 1]$, Bayesian incentive compatible.

Outline

In quasilinear setting, a simple mechanism is DSE and maximizes the social welfare $\sum_i v_i(\omega)$

In quasilinear setting, a simple mechanism is DSE and maximizes the social welfare $\sum_i v_i(\omega)$

Vickrey Clarke Groves (VCG) Mechanism

1. Solicit type v_i from each player i
2. Choose allocation $\omega^* \in \text{argmax}_{\omega \in \Omega} \sum_i v_i(\omega)$
3. Charge each player i payment $p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j(\omega^*)$

- Allocation rule maximizes welfare exactly over Ω
- Player i is paid the reported value of others for the chosen allocation, less a pivot term $h_i(v_{-i})$ independent of his own bid.

In quasilinear setting, a simple mechanism is DSE and maximizes the social welfare $\sum_i v_i(\omega)$

Vickrey Clarke Groves (VCG) Mechanism

1. Solicit type v_i from each player i
2. Choose allocation $\omega^* \in \operatorname{argmax}_{\omega \in \Omega} \sum_i v_i(\omega)$
3. Charge each player i payment $p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j(\omega^*)$

- Allocation rule maximizes welfare exactly over Ω
- Player i is paid the reported value of others for the chosen allocation, less a pivot term $h_i(v_{-i})$ independent of his own bid.
- In most cases, the "right" pivot term is $\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega)$
  - Payment $p_i(v)$ is player i's is externality
  - $0 \leq p_i(v) \leq v_i(\omega^*)$

VCG is Truthful

Theorem

VCG is dominant-strategy truthful.

Proof

- Fix reports v_{-i} of players other than i.
- Assume player i's true valuation is v_i
- Player i's utility when reporting $\widehat{v}_i$ is given by

$$u_i(\widehat{v}_i) = v_i(\omega^*) + \sum_{j \neq i} v_j(\omega^*) - h_i(v_{-i}),$$

  where $\omega^* \in \operatorname{argmax}_{\omega \in \Omega} \left(\widehat{v}_i(\omega) + \sum_{j \neq i} v_j(\omega)\right)$
- Since the pivot term is independent of player i's bid, maximizing $u_i(\widehat{v}_i)$ is equivalent to maximizing

$$v_i(\omega^*) + \sum_{j \neq i} v_j(\omega^*)$$

- Setting $\widehat{v}_i = v_i$ then maximizes the above expression.
    - Interpretation: allow the mechanism to optimize player i's utility on his behalf

Example: Single-item Allocation

- Welfare maximizing outcome: Allocate to player with highest value

Example: Single-item Allocation

- Welfare maximizing outcome: Allocate to player with highest value
- Externality of i: second-highest value if i wins, 0 otherwise.

Example: Single-item Allocation

- Welfare maximizing outcome: Allocate to player with highest value
- Externality of i: second-highest value if i wins, 0 otherwise.

VCG is the second-price (Vickrey) auction in the special case of single-item allocation.

Outline

Maximizing Revenue

Well understood in the case of single-parameter problems

Single-parameter problem (informally)

- There is a single homogenous resource.
- Constraints on how much of the resource each player can get
- Each player's type is his "value (or cost) per unit resource."

Maximizing Revenue

Well understood in the case of single-parameter problems

Single-parameter problem (informally)

- There is a single homogenous resource.
- Constraints on how much of the resource each player can get
- Each player's type is his "value (or cost) per unit resource."

Canonical example: single-item allocation

- Resource: one unit of item
- Outcomes Ω: vectors $(x_1, \ldots, x_n)$ with $x_i \geq 0$ and $\sum_i x_i \leq 1$
  - x_i is probability player i gets item
- Player i's type is $v_i \geq 0$ (value for item)
  - $u_i(x, p) = v_i x_i - p_i$

Maximizing Revenue

Makes most sense in Bayesian setting with independent types (prior $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ on $(v_1, \ldots, v_n)$)

Bayesian Revenue Maximization (Single Parameter)

Given prior $\mathcal{F}$ on type profiles $T \subseteq \mathbb{R}^n$, find allocation rule $x : T \to \Omega$ (recall $\Omega \subseteq \mathbb{R}^n$) and payment rules $p : T \to \mathbb{R}^n$ such that

- (x, p) is a BIC direct revelation mechanism
  - Bidding $b_i = v_i$ maximizes $\mathbf{E}_{v_{-i} \sim \mathcal{F}_{-i}}[v_i x_i(b_i, v_{-i}) - p_i(b_i, v_{-i})]$
- $Rev(x, p) = \mathbf{E}_{v \sim \mathcal{F}} \sum_i p_i(v)$ is as large as possible.

Maximizing Revenue

Makes most sense in Bayesian setting with independent types (prior $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ on $(v_1, \ldots, v_n)$)

Bayesian Revenue Maximization (Single Parameter)

Given prior $\mathcal{F}$ on type profiles $T \subseteq \mathbb{R}^n$, find allocation rule $x : T \to \Omega$ (recall $\Omega \subseteq \mathbb{R}^n$) and payment rules $p : T \to \mathbb{R}^n$ such that

- (x, p) is a BIC direct revelation mechanism
  - Bidding $b_i = v_i$ maximizes $\mathbf{E}_{v_{-i} \sim \mathcal{F}_{-i}}[v_i x_i(b_i, v_{-i}) - p_i(b_i, v_{-i})]$
- $Rev(x, p) = \mathbf{E}_{v \sim \mathcal{F}} \sum_i p_i(v)$ is as large as possible.

Myerson characterized the optimal solution for single-item auctions, and it generalizes easily to single-parameter environments

- Think of single-item auctions in upcoming discussion

Stages of a Bayesian Game

Stages of a Bayesian game of mechanism design:

- Ex-ante: Before players learn their types
- Interim: A player learns his type, but not the types of others.
- Ex-post All player types are revealed.

Stages of a Bayesian Game

Stages of a Bayesian game of mechanism design:

- Ex-ante: Before players learn their types
- Interim: A player learns his type, but not the types of others.
- Ex-post All player types are revealed.

Interim stage is when players make decisions.

- The interim allocation rule for player i tells us what the probability of winning (expected amount of resource) is as a function of player i's bid, in expectation over other player's truthful reports.

$$\overline{x}_i(b_i) = \mathop{\mathbf{E}}_{v_{-i} \sim \mathcal{F}_{-i}} [x_i(b_i, v_{-i})]$$

- Similarly, the interim payment rule.

$$\overline{p}_i(b_i) = \mathop{\mathbf{E}}_{v_{-i} \sim \mathcal{F}_{-i}} [p_i(b_i, v_{-i})]$$

Stages of a Bayesian Game

Stages of a Bayesian game of mechanism design:

- Ex-ante: Before players learn their types
- Interim: A player learns his type, but not the types of others.
- Ex-post All player types are revealed.

Interim stage is when players make decisions.

- The interim allocation rule for player i tells us what the probability of winning (expected amount of resource) is as a function of player i's bid, in expectation over other player's truthful reports.

$$\overline{x}_i(b_i) = \mathop{\mathbf{E}}_{v_{-i} \sim \mathcal{F}_{-i}} [x_i(b_i, v_{-i})]$$

- Similarly, the interim payment rule.

$$\overline{p}_i(b_i) = \mathop{\mathbf{E}}_{v_{-i} \sim \mathcal{F}_{-i}} [p_i(b_i, v_{-i})]$$

- BIC: Bidding $b_i = v_i$ maximizes $v_i \overline{x}_i(b_i) - \overline{p}_i(b_i)$

Stages of a Bayesian Game

Stages of a Bayesian game of mechanism design:

- Ex-ante: Before players learn their types
- Interim: A player learns his type, but not the types of others.
- Ex-post All player types are revealed.

Interim stage is when players make decisions.

- The interim allocation rule for player i tells us what the probability of winning (expected amount of resource) is as a function of player i's bid, in expectation over other player's truthful reports.

$$\overline{x}_i(b_i) = \mathop{\mathbf{E}}_{v_{-i} \sim \mathcal{F}_{-i}} [x_i(b_i, v_{-i})]$$

- Similarly, the interim payment rule.

$$\overline{p}_i(b_i) = \mathop{\mathbf{E}}_{v_{-i} \sim \mathcal{F}_{-i}} [p_i(b_i, v_{-i})]$$

- BIC: Bidding $b_i = v_i$ maximizes $v_i \overline{x}_i(b_i) - \overline{p}_i(b_i)$
- If BIC, then $Rev(x, p) = \sum_i \mathbf{E}_{v_i \sim F_i} \overline{p}(v_i)$

Assume two players drawn independently from $U[0,1]$.

Vickrey Auction

- $\overline{x}_i(v_i) = v_i$
- $\overline{p}_i(v_i) = v_i^2/2$.

Assume two players drawn independently from $U[0,1]$.

Vickrey Auction

- $\overline{x}_i(v_i) = v_i$
- $\overline{p}_i(v_i) = v_i^2/2$.

First Price Auction

- $\overline{x}_i(v_i) = v_i$
- $\overline{p}_i(v_i) = v_i^2/2$

Assume two players drawn independently from $U[0,1]$.

Vickrey Auction

- $\overline{x}_i(v_i) = v_i$
- $\overline{p}_i(v_i) = v_i^2/2$.

First Price Auction

- $\overline{x}_i(v_i) = v_i$
- $\overline{p}_i(v_i) = v_i^2/2$

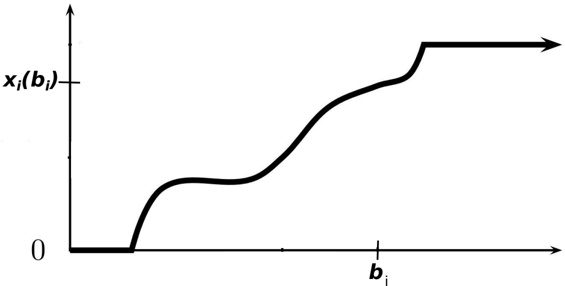
From now on we will write $x_i(b_i) = \overline{x}_i(b_i)$ to avoid cumbersome notation

Myerson's Monotonicity Lemma

Consider a mechanism for a single-parameter problem in a Bayesian setting where player values are independent. A direct-revelation mechanism with interim allocation rule x and payment rule p is BIC if and only if for each player i:

- $x_i(b_i)$ is a monotone non-decreasing function of b_i
- $p_i(b_i)$ is an integral of $b_i \; dx_i$. Specifically, when $p_i(0) = 0$ then

$$p_i(b_i) = b_i \cdot x_i(b_i) - \int_{b=0}^{b_i} x_i(b)db$$

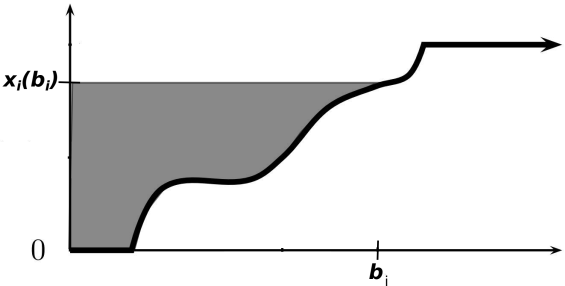


Myerson's Monotonicity Lemma

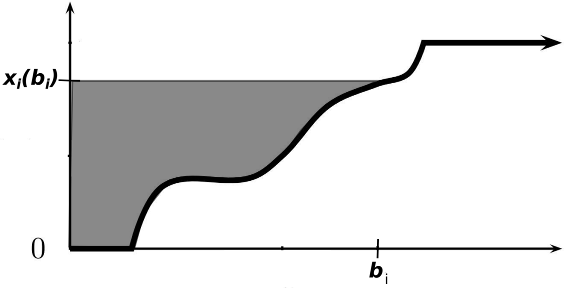
Consider a mechanism for a single-parameter problem in a Bayesian setting where player values are independent. A direct-revelation mechanism with interim allocation rule x and payment rule p is BIC if and only if for each player i:

- $x_i(b_i)$ is a monotone non-decreasing function of b_i
- $p_i(b_i)$ is an integral of $b_i\ dx_i$. Specifically, when $p_i(0) = 0$ then

$$p_i(b_i) = b_i \cdot x_i(b_i) - \int_{b=0}^{b_i} x_i(b)db$$

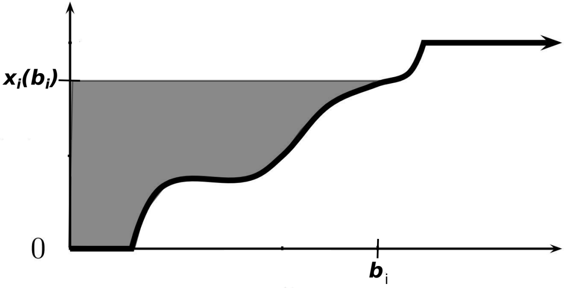


Interpretation of Myerson's Monotonicity Lemma



- The higher a player bids, the higher the probability of winning.
- For each additional sliver ϵ of winning probability, pays at a rate equal to the minimum bid needed to acquire that sliver
  - Recall: second price auction

Interpretation of Myerson's Monotonicity Lemma



- The higher a player bids, the higher the probability of winning.
- For each additional sliver ϵ of winning probability, pays at a rate equal to the minimum bid needed to acquire that sliver
  - Recall: second price auction

See readings for proof of Myerson's monotoncity Lemma

Corollaries of Myerson's Monotonicity Lemma

Corollaries

- Interim allocation rule uniquely determines interim payment rule.
- Expected revenue depends only on the allocation rule

Theorem (Revenue Equivalence)

Any two auctions with the same interim allocation rule in BNE have the same expected revenue in the same BNE.

Revenue as Virtual Welfare

Define the virtual value of player i as a function of his value v_i

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

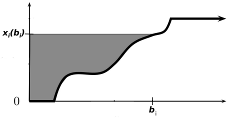
Lemma (Myerson's Virtual Welfare Lemma)

Consider a BIC mechanism M with interim allocation rule x and payment rule p, and assume that $p_i(0) = 0$ for all i. The expected revenue of M is equal to the expected virtual welfare served.

$$\sum_i \mathop{\mathbf{E}}_{v_i \sim F_i}[\phi(v_i) x(v_i)]$$

In single-item auction, this is the expected virtual value of the winning bidder.

Proof



$$
\begin{aligned}
\mathop{\mathbf{E}}_{v\sim\mathcal{F}_i}[p_i(v)] &= \int_v \left[v x_i(v) - \int_{b=0}^{v} x_i(b)db \right] f_i(v)dv \\
&= \int_v v x_i(v) f_i(v) dv - \int_v \int_{b\le v} x_i(b) f_i(v) db dv \\
&= \int_v v x_i(v) f_i(v) dv - \int_b x_i(b) \int_{v\ge b} f_i(v) dv db \\
&= \int_v v x_i(v) f_i(v) dv - \int_b x_i(b)(1 - F_i(b)) db \\
&= \int_v \left[v x_i(v) f_i(v) - x_i(v)(1 - F_i(v)) \right] dv \\
&= \int_v f_i(v) x_i(v) \left[v - \frac{1 - F_i(v)}{f_i(v)} \right] dv = \int_v f_i(v) x_i(v) \phi_i(v_i) dv
\end{aligned}
$$

Myerson's Revenue-Optimal Auction

1. Solicit player values
2. If at least one player has nonnegative virtual value, then give the item to the player i with the highest virtual value $\phi_i(v_i) \geq 0$. Otherwise, nobody gets the item.
3. Charge the minimum bid needed to win $\phi_i^{-1}(\max(0, (\max_{j \neq i} \phi_j(v_j))))$
   - Check: satisfies Myerson's condition on interim payment

Myerson's Revenue-Optimal Auction

1. Solicit player values
2. If at least one player has nonnegative virtual value, then give the item to the player i with the highest virtual value $\phi_i(v_i) \geq 0$. Otherwise, nobody gets the item.
3. Charge the minimum bid needed to win $\phi_i^{-1}(\max(0, (\max_{j \neq i} \phi_j(v_j))))$
   - Check: satisfies Myerson's condition on interim payment

Observations

- The allocation rule maximizes virtual welfare point-wise
- Therefore, it maximizes expected virtual welfare over all allocation rules.
- By Myerson's virtual welfare Lemma, revenue is at least that of any BIC mechanism (since any BIC mechanism's revenue is equal to expected virtual welfare).

Myerson's Revenue-Optimal Auction

1. Solicit player values
2. If at least one player has nonnegative virtual value, then give the item to the player i with the highest virtual value $\phi_i(v_i) \geq 0$. Otherwise, nobody gets the item.
3. Charge the minimum bid needed to win $\phi_i^{-1}(\max(0, (\max_{j \neq i} \phi_j(v_j))))$
   - Check: satisfies Myerson's condition on interim payment

Observations

- The allocation rule maximizes virtual welfare point-wise
- Therefore, it maximizes expected virtual welfare over all allocation rules.
- By Myerson's virtual welfare Lemma, revenue is at least that of any BIC mechanism (since any BIC mechanism's revenue is equal to expected virtual welfare).

Are we done?

A Wrinkle

Not really... What if the allocation rule of the mechanism we just defined is non-monotone? It would still have revenue at least that of the optimal BIC mechanism if players happened to report truthfully, but it wouldn't be truthful itself

A Wrinkle

Not really... What if the allocation rule of the mechanism we just defined is non-monotone? It would still have revenue at least that of the optimal BIC mechanism if players happened to report truthfully, but it wouldn't be truthful itself

Fortunately

Virtual welfare maximization is monotone when the distributions are regular!!

- $\phi_i(v) = v - \frac{1-F_i(v)}{f_i(v)}$ is nondecreasing in v

Conclude

When distributions are regular, the VV maximizing auction (aka Myerson's optimal auction) is the revenue-optimal BIC mechanism!

A Wrinkle

Not really... What if the allocation rule of the mechanism we just defined is non-monotone? It would still have revenue at least that of the optimal BIC mechanism if players happened to report truthfully, but it wouldn't be truthful itself

Fortunately

Virtual welfare maximization is monotone when the distributions are regular!!

- $\phi_i(v) = v - \frac{1-F_i(v)}{f_i(v)}$ is nondecreasing in v

Conclude

When distributions are regular, the VV maximizing auction (aka Myerson's optimal auction) is the revenue-optimal BIC mechanism!

- Most natural dists are regular (Gaussian, uniform, exp, etc).
- Can be extended to non-regular distributions via ironing, which we will not discuss now (if at all).

Thoughts

Myerson's optimal auction is noteworth for many reasons

- Matches practical experience: when players i.i.d regular, optimal auction is Vickrey with reserve price $\phi^{-1}(0)$.
- Applies to single parameter problems more generally
- Revenue maximization reduces to welfare maximization for these problems
- The optimal BIC mechanism just so happens to be DSIC and deterministic!!