2/2/2010

Randomization

-can be used to improve algorithms

Example: Testing for prime numbers using coin flips

Sorting

Quick Sort

① choose first element and treat it as the ~~median~~ pivot,

② make two groups

- one greater than the pivot
- one less than the pivot

③ recurse with each new group

Perfect split:

$$T(32) = 2T(16) + 32$$

Less Perfect split:

$$T(32) = T(6) + T(24) + 32$$

$$T(32) = 1 + T(31) + 32$$

$$\downarrow$$

$$1 + T(31) + 31$$

$$\Omega(n^2)$$

How can we better choose the pivot?

Randomness

coin $\in \{H, T\}$
$\{0, 1\}$

Probability [coin = "H"] $= \frac{1}{2} =$ Pr [coin = "T"]
("Perfect Coin")

K-dice $\{0, 1, \cdots, k-1\}$
Pr [K-dice $= i$] $= \frac{1}{k}$ (uniform dice)
- all outcomes have equal probability

n-elements $\underbrace{\text{2-dice 3-dice} \;\ldots\; \text{n-dice}}$

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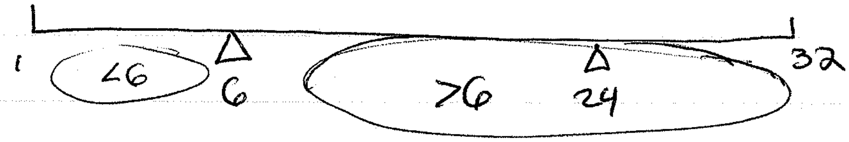
selection Problem

INPUT: $A[1], A[2], \cdots\cdots, A[n]$ - array of #s
k - integer
OUTPUT: kth smallest element of $A[i] \cdots$

How quickly can we solve this problem using randomness?

Suppose array has 1 - 32 elements
looking for 24th



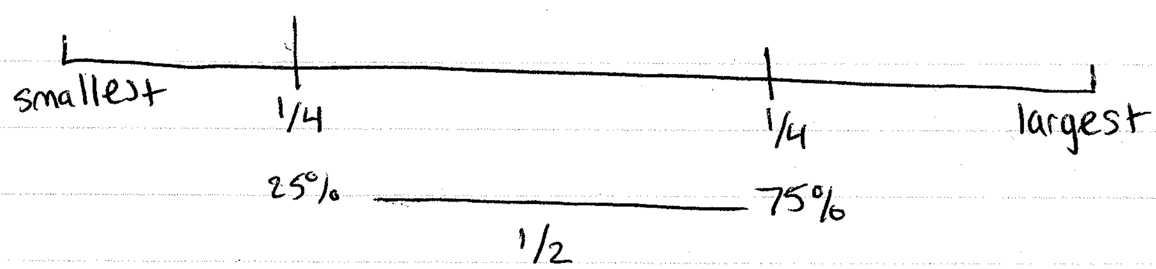
- randomly choose 6,
- use as pivot and sort into 2 groups, <6 and >6
- since 24 is >6, throw away <6 group
- recurse, using >6 group

$$T(32) = T(26) + 32$$

randomness useful in selecting pivot

$$E\left(T_{RS}(n)\right) = O(n)$$

$$E\left(\text{Iteration}(n)\right) = O(\log(n))$$



If x is random from $[1, n]$ uniformly:

$$\Pr\left[\frac{n}{4} \le x \le \frac{3}{4}n\right] = \frac{1}{2}$$

$$E\left(T_{TRS}(n)\right) = \frac{1}{2}E\left(T_{TRS}\left(\frac{3}{4}n\right)\right) +$$

$\boxed{RS = \text{Random selection}}$

$$\frac{1}{2}E\left(T_{TRS}(n)\right) + n$$

$$E\left(T_{TRS}(n)\right) \leq E\left(T_{TRS}\left(\frac{3}{4}n\right)\right) + 2n$$

$$\downarrow$$

$$E\left(T_{TRS}\left(\left(\frac{3}{4}\right)^2 n\right)\right) + 2\cdot\frac{3}{4}n + 2n$$

$$\downarrow$$

$$= 1 \cdots + 2n\left(\frac{3}{4}\right)^3 + 2n\left(\frac{3}{4}\right)^2 + 2n\left(\frac{3}{4}\right) + 2n$$

$$= \Theta(n)$$

$$E\left(T_{QSORT}(n)\right) = \Theta(n \log n)$$

Birthday Paradox –

suppose: a year has n days
m people with random birthdays

Pr[two of them has the same birthday] (collision)

if $m=1$ — $Pr[m,n]=0$

if $m=2$ — $Pr[2,n] = \frac{1}{n}$

if $m=3$ — $Pr[3,n] = \frac{\quad}{n^3}$

3way collisions $\qquad$ # 2-way collisions

$$\frac{n}{n^3} + \frac{3n(n-1)}{n^3}$$

= number of 3-way collision

$m = 100$

$$\binom{m}{2} = \frac{m(m-1)}{2} \text{ pairs}$$

$$Pr[m,n] \geq \frac{\frac{m(m-1)}{2} \cdot n \cdot n^{m-2}}{n^m}$$

$$= \frac{m(m-1)}{2n}$$