

Shadow prices in territory division

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Abstract

We consider a geographic optimization problem in which we are given a region R , a probability density function $f(\cdot)$ defined on R , and a collection of n utility density functions $u_i(\cdot)$ defined on R . Our objective is to divide R into n sub-regions R_i so as to “balance” the overall utilities on the regions, which are given by the integrals $\iint_{R_i} f(x)u_i(x) dA$. Using a simple complementary slackness argument, we show that (depending on what we mean precisely by “balancing” the utility functions) the boundary curves between optimal sub-regions are level curves of either the difference function $u_i(x) - u_j(x)$ or the ratio $u_i(x)/u_j(x)$. This allows us to solve the problem of optimally partitioning the region efficiently by reducing it to a low-dimensional convex optimization problem. This result generalizes, and gives very short and constructive proofs of, several existing results in the literature on equitable partitioning for particular forms of $f(\cdot)$ and $u_i(\cdot)$. We next give two economic applications of our results in which we show how to compute a market-clearing price vector in an *aggregate demand system* or a variation of the classical *Fisher exchange market*. Finally, we consider a *dynamic* problem in which the density function $f(\cdot)$ varies over time (simulating population migration or transport of a resource, for example) and derive a set of partial differential equations that describe the evolution of the optimal sub-regions over time. Numerical simulations for both static and dynamic problems confirm that such partitioning problems become tractable when using our methods.

1 Introduction

Dividing a given territory into pieces is a fundamental geographic problem with many application areas, including logistics, economics, and natural resource allocation. A simple mathematical formulation of such a problem is as follows: suppose that R is a geographic region in the plane which we are to partition among n agents, that is, we are to select n sub-regions R_1, \dots, R_n of R such that $R_i \cap R_j = \emptyset$ for all pairs and $\bigcup_i R_i = R$. In this paper, we will assume that R is a connected, polygonal region with non-empty interior. Letting $u_i(\cdot)$ denote a “utility density” function associated with agent i , we can represent the overall utility of agent i as the integral $\iint_{R_i} u_i(x) dA$, where R_i denotes the sub-region assigned to agent i . In order to generalize this model further, let us also assume that $f(\cdot)$ is a given probability density function (representing population or distribution of a natural resource, for example) on R , so that the overall utility of agent i is the integral $\iint_{R_i} f(x)u_i(x) dA$. This problem has been previously considered in many different domains for various particular forms of $f(\cdot)$ and $u_i(\cdot)$ which we will discuss shortly.

The key issue in the preceding problem is how to construct the sub-regions in a *balanced*, or *equitable*, fashion. One way to partition the region is to maximize the overall utilities of the agents while imposing constraints on the amounts of $f(\cdot)$ that are contained in them. That is, our problem can be written as

$$\begin{aligned} \underset{R_1, \dots, R_n}{\text{maximize}} \quad & \sum_{i=1}^n \iint_{R_i} f(x)u_i(x) dA \quad \quad \quad s.t. \\ & \iint_{R_i} f(x) dA = q_i \quad \forall i \\ & R_i \cap R_j = \emptyset \quad \forall i \neq j \\ & \bigcup_i R_i = R \end{aligned} \tag{1}$$

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where the q_i are given constants. A second way to partition the region in a balanced way is to maximize the *minimum* utility of all of the sub-regions:

$$\begin{aligned} \text{maximize } \min_{R_1, \dots, R_n} \left\{ \iint_{R_i} f(x) u_i(x) dA \right\} \quad & s.t. \\ R_i \cap R_j &= \emptyset \quad \forall i \neq j \\ \bigcup_i R_i &= R. \end{aligned} \quad (2)$$

In this paper we show that the boundaries between the optimal sub-regions to problem (1) are curves of the form

$$x : u_i(x) - u_j(x) = \text{constant}$$

and that the boundaries between the optimal sub-regions to problem (2) are curves of the form

$$x : \frac{u_i(x)}{u_j(x)} = \text{constant}$$

provided that either $u_i(x) > 0$ for all i and x or $u_i(x) < 0$ for all i and x (except for possibly a set of measure zero). Although this turns out to be a simple and immediate consequence of complementary slackness in vector space optimization, it allows us to obtain very concise, constructive proofs to well-known existing results in equitable partitioning. Moreover, our proof technique reduces both balanced partitioning problems to n -dimensional convex optimization problems, which allows us to actually solve both problems efficiently in practice by computing a set of shadow prices associated with the agents. The remaining contributions of this paper are as follows: in Section 4, we give fast algorithms for solving (1) and (2) by showing how to compute a subgradient vector for either problem, which enables us to use (for example) a *cutting plane method* to find the optimal partition. Section 6 gives two economic applications of these principles to compute a market-clearing price vector in an *aggregate demand system* or a variation of the classical *Fisher exchange market*. Section 9 then considers a *dynamic* problem in which the density function $f(\cdot)$ varies over time (simulating population migration or transport of a resource, for example) and we derive a set of partial differential equations that describe the evolution of the optimal sub-regions over time.

Related work

Problems (1) and (2) have already been studied for specific forms of the functions $u_i(\cdot)$. The case of problem (1) where $u_i(x) = -\|x - p_i\|^2$ for fixed points $p_i \in R$ was first analyzed in [5] and later in [33, 34]; the former gives a fast algorithm for optimal partitioning for the case where $f(\cdot)$ is an atomic distribution and the latter gives a control scheme that converges to an optimal partition for smooth $f(\cdot)$. The case of (1) where $u_i(x) = -\|x - p_i\|$ and $f(\cdot)$ is a uniform distribution was analyzed in [3] (whose analysis also extends cleanly to general distributions $f(\cdot)$) who also give an approximation algorithm for simultaneously locating the points p_i and designing partitions.

Our recent paper [12] considers the special case of problems (1) and (2) where $u_i(x) = -\|x - p_i\|$, where we derive solutions for the optimal partition using complementary slackness techniques.

Notational conventions

In this paper we adopt the standard notation of vector calculus; in particular, we let $\partial(\cdot)$ denote the *boundary operator* and we let $\nabla \cdot$ denote the *divergence operator*. We use a double integral $\iint_R f(x) dA$ to denote integration over a planar region whereas we use a triple integral $\iiint_R f(x) dV$ to denote integration over a domain of arbitrary dimension.

2 Applications

Before discussing the solution method for problems (1) and (2), we first give several geographic applications thereof.

Power diagrams A *power diagram* of a set of points $\{p_1, \dots, p_n\} \subset \mathbb{R}^d$ is a generalization of the well-known *Voronoi diagram* in which additive weights w_i are associated with each of the points. Specifically, an arbitrary point $x \in \mathbb{R}^d$ is said to belong to the *power cell* C_i of the point p_i if

$$\|x - p_i\|^2 - w_i \leq \|x - p_j\|^2 - w_j$$

for all indices $j \in \{1, \dots, n\}$. It turns out that the cells C_i in any power diagram are always convex. Power diagrams are widely applied in such diverse domains as robotics [33, 34, 35], air traffic control [14], and sensor placement [16]. The paper [5] proves that the optimal solution to problem (1) is always a power diagram whenever $u_i(x) = -\|x - p_i\|^2$, i.e. when our objective is to minimize the sum of the mass moments of the sub-regions. An immediate corollary (which was independently re-derived in [33, 34]) is that, given any set of points $\{p_1, \dots, p_n\}$ in a region R , a probability density $f(\cdot)$ defined on R , and any positive vector (q_1, \dots, q_n) whose entries sum to 1, there always exists a set of weights whose resulting power diagram satisfies $\iint_{C_i} f(x) dA = q_i$ for each i .

Fair division In a *fair division* problem we are concerned with dividing the region R “fairly” [9]. One interpretation of fairness is precisely the formulation (2) with $f(x) = 1$ everywhere and generic functions $u_i(\cdot)$. It turns out (as we will show momentarily) that the optimal solution to (2) has equal values of $\iint_{R_i} f(x) u_i(x) dA$ (i.e. $\iint_{R_i} u_i(x) dA$ in our case) for all i ; thus, each agent’s utility is equal at optimality.

Maximizing influence The *gravity hypothesis* [39] is a well-known geographic theory that states that the “interaction” between two points x and y generally decays at a rate proportional to the inverse square of the distance between them, i.e. $1/\|x - y\|^2$. Here “interaction” might be measured by economic activity [6], migration [24], or transport [38], for example. It follows that if p_i is the “capital” of region R_i then the total “influence” (economic, political, or cultural) that p_i exercises over R_i can be approximated as

$$\iint_{R_i} \frac{f(x)}{\|x - p_i\|^2} dA$$

where $f(\cdot)$ represents a population density, so that $u_i(x) = 1/\|x - p_i\|^2$. Since the integral blows up near the points p_i it is natural to truncate $u_i(x)$ in a small ϵ -neighborhood of p_i . A natural application of problems (1) and (2) arises in the division of territory among a collection of “capital cities” $\{p_1, \dots, p_n\}$ so as to maximize the influence that the cities exercise over their respective domains R_i while respecting overall constraints on the populations of these domains. The related *geographic potential model* [40] postulates that the interaction between two points decays at a rate directly proportional to the inverse of the distance between them, which gives $u_i(x) = 1/\|x - p_i\|$.

Hospital districting It has been observed among geographers that, in rural regions, the frequency of a person’s visits to a hospital decays exponentially in their distance to the hospital [19, 41]. Thus, if a hospital located at a point p_i provides service to a region R_i , the long-term workload that the hospital experiences can roughly be approximated by

$$\iint_{R_i} f(x) \exp(-\|x - p_i\|) dA$$

where $f(\cdot)$ represents a population density. Since the goal of districting among hospitals is generally to ensure equity among the service regions, it is therefore natural to consider problem (2) with $u_i(x) = \exp(-\|x - p_i\|)$. Exponential distance decay also occurs in measuring ecological similarity between areas [30, 36]; in this case, problem (2) determines a partition of R into various sub-regions whose overall similarity with test sites p_i is maximized.

Facility districting In the recent paper [12] we have considered the problem of dividing a territory among a collection of facilities $\{p_1, \dots, p_n\}$ so as to balance the workloads of those facilities, where we model the workload of a facility covering region R_i as

$$\iint_{R_i} f(x) \|x - p_i\| dA.$$

The simple intuition is that the cost of p_i providing service to a point x is simply proportional to the distance between x and p_i . Thus here we can consider either problem (1) or (2) with $u_i(x) = -\|x - p_i\|$. We can also extend this to consider the case where each agent i has a *collection* of facilities $\{p_1^i, \dots, p_{n_i}^i\}$ (representing multiple branches of a store, for example), which gives $u_i(x) = -\min_{j \in \{1, \dots, n_i\}} \|x - p_j^i\|$.

When the region R contains obstacles (or “holes”), it may not be appropriate to measure the distance between a point x and a facility p_i under the Euclidean norm. Instead, we may use the distance function $d(x, p_i)$ which gives the length of the shortest path between x and p_i .

Districts for vehicle routing In a *vehicle districting problem* our objective is to design a collection of sub-regions that minimize the workloads of a fleet of vehicles that provide service to a region [18]. Suppose that, on each day, a set of N demand points $\{X_1, \dots, X_N\}$ is sampled from a probability distribution $\tilde{f}(\cdot)$ defined on a service region R , and each demand point must be serviced by a vehicle (each of which is originally located at a depot $p_i \in R$). The workload of a vehicle assigned to cover region R_i is simply the length of a travelling salesman tour of all of the sampled points in R_i plus the depot point p_i , $\text{TSP}(p_i \cup \{X_j\} \cap R_i)$. As we have shown previously in [11], it turns out that with probability one we have

$$\text{TSP}(p_i \cup \{X_j\} \cap R_i) \rightarrow \beta \iint_{R_i} \sqrt{\tilde{f}_c(x)} dA + o(\sqrt{N})$$

as $N \rightarrow \infty$, where $\tilde{f}_c(\cdot)$ denotes the absolutely continuous part of $\tilde{f}(\cdot)$. In order to balance the workloads of the vehicles, we should then solve an instance of (1) where $f(\cdot) = \sqrt{\tilde{f}_c(x)}$ (normalized so that $f(\cdot)$ integrates to 1) and $q_i = 1/n$, so that all vehicles have the same asymptotic workload. As a utility function it is natural to use $u_i(x) = -\|x - p_i\|$ or $u_i(x) = -\|x - p_i\|^2$ so that our sub-regions are as “compact” as possible. We will see that in fact the optimal sub-regions will always be connected when such functions are chosen.

Police dragnet design It has been hypothesized [10, 32] that a *logarithmic* relationship exists between the distance from a criminal’s home base to a potential target location and the likelihood that the offender chooses to offend in that location, i.e. that

$$\Pr(\text{criminal strikes at } x | \text{home base at } y) = \max\{a - b \log \|x - y\|, 0\}$$

with $a, b \geq 0$. It follows that if a crime has occurred at a point p_i then the probability that the criminal’s home base is located in region R_i is proportional to

$$\iint_{R_i} f(x) \max\{a - b \log \|x - p_i\|, 0\} dA$$

where $f(x)$ represents a population density. Thus, given a collection of recent crime locations $\{p_1, \dots, p_n\}$, one can thus consider the problem of designing police “search regions” to maximize the likelihood of catching the criminals by formulating problem (1) with $u_i(x) = \max\{a - b \log \|x - p_i\|, 0\}$. Here we might set $q_i = 1/n$ in the constraint $\iint_{R_i} f(x) dA = q_i$ for all i , representing a restriction that each district have an equal population (and thus roughly equal “workloads” for the police investigators).

3 Optimal sub-regions for (1) and (2)

In this section we show that the optimal sub-regions for problems (1) and (2) can be described easily in terms of complementary slackness. For ease of intuition, we give proof sketches based on discretizing the problem; the same result can be derived rigorously using infinite-dimensional vector space optimization theory, specifically Theorem 1 of [26], and we do so in Section A of the online supplement.

3.1 Sub-regions for (1)

In this section we consider the structure of problem (1). We provide a proof sketch that characterizes the optimal solutions and refer the reader to Section A.1 of the online supplement for a rigorous proof:

Theorem 1. *The boundaries between any optimal sub-regions R_i^* and R_j^* to problem (1) are of the form*

$$\partial(R_i^*) \cap \partial(R_j^*) \subseteq \{x \in R : u_i(x) - u_j(x) = \lambda_i^* - \lambda_j^*\}$$

where λ_i^* and λ_j^* are the optimal solutions to the dual problem

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & \iint_R f(x) \max_i \{u_i(x) - \lambda_i\} dA \quad \text{s.t.} \\ & \mathbf{q}^T \boldsymbol{\lambda} = 0. \end{aligned}$$

Moreover, if x and i are a point and an index such that $u_i(x) - \lambda_i^* > u_j(x) - \lambda_j^*$ for all $j \neq i$, then $x \in R_i^*$. Thus, the optimal partition $\{R_1^*, \dots, R_n^*\}$ can be recovered from the optimal dual variables $\lambda_1^*, \dots, \lambda_n^*$.

Proof sketch. We begin by formulating problem (1) as an infinite-dimensional integer program. Setting $I_i(x)$ to be a $\{0, 1\}$ -valued function indicating whether point x is assigned to agent i , we obtain the equivalent formulation

$$\begin{aligned}
& \underset{I_1(\cdot), \dots, I_n(\cdot)}{\text{maximize}} \sum_{i=1}^n \iint_R f(x) u_i(x) I_i(x) dA && s.t. \\
& \iint_R f(x) I_i(x) dA = q_i && \forall i \\
& \sum_{i=1}^n I_i(x) = 1 && \forall x \\
& I_i(x) \in \{0, 1\} && \forall i, x.
\end{aligned} \tag{3}$$

The linear programming relaxation of (3) is given by

$$\begin{aligned}
& \underset{I_1(\cdot), \dots, I_n(\cdot)}{\text{maximize}} \sum_{i=1}^n \iint_R f(x) u_i(x) I_i(x) dA && s.t. \\
& \iint_R f(x) I_i(x) dA = q_i && \forall i \\
& \sum_{i=1}^n I_i(x) = 1 && \forall x \\
& I_i(x) \geq 0 && \forall i, x.
\end{aligned} \tag{4}$$

We can discretize the above problem into N grid cells \square_j of area ϵ , where f_j denotes the average value of $f(x)$ on \square_j , u_{ij} denotes the average value of $u_i(x)$ on \square_j , and z_{ij} denotes the fraction of cell \square_j assigned to agent i , to obtain the approximate formulation

$$\begin{aligned}
& \underset{Z}{\text{maximize}} \sum_{i=1}^n \sum_{j=1}^N \epsilon f_j u_{ij} z_{ij} && s.t. \\
& \sum_{j=1}^N \epsilon f_j z_{ij} = q_i && \forall i \\
& \sum_{i=1}^n z_{ij} = 1 && \forall j \\
& z_{ij} \geq 0 && \forall i, j.
\end{aligned} \tag{5}$$

The dual problem to (5), which has variables $\lambda \in \mathbb{R}^n$ and $\varsigma \in \mathbb{R}^N$, is

$$\begin{aligned}
& \underset{\lambda, \varsigma}{\text{minimize}} \sum_{i=1}^n q_i \lambda_i + \sum_{j=1}^N \varsigma_j && s.t. \\
& \epsilon f_j \lambda_i + \varsigma_j \geq \epsilon f_j u_{ij} && \forall i, j.
\end{aligned}$$

Introducing new variables $\sigma_j := \varsigma_j / (\epsilon f_j)$, we can rewrite the above as

$$\begin{aligned}
& \underset{\lambda, \sigma}{\text{minimize}} \sum_{i=1}^n q_i \lambda_i + \sum_{j=1}^N \epsilon f_j \sigma_j && s.t. \\
& \sigma_j \geq u_{ij} - \lambda_i && \forall i, j
\end{aligned}$$

which is a discretization of the problem

$$\begin{aligned}
& \underset{\lambda, \sigma(\cdot)}{\text{minimize}} \sum_{i=1}^n q_i \lambda_i + \iint_R f(x) \sigma(x) dA && s.t. \\
& \sigma(x) \geq u_i(x) - \lambda_i && \forall i, x
\end{aligned} \tag{6}$$

which is equivalent to the unconstrained problem

$$\underset{\boldsymbol{\lambda}}{\text{minimize}} \sum_{i=1}^n q_i \lambda_i + \iint_R f(x) \max_i \{u_i(x) - \lambda_i\} dA.$$

Finally, we note that the above problem is invariant under scalar addition to $\boldsymbol{\lambda}$ because we have assumed that $\sum_{i=1}^n q_i = \iint_R f(x) dA = 1$ and thus we obtain the convex, n -dimensional dual problem

$$\begin{aligned} \underset{\boldsymbol{\lambda}}{\text{minimize}} \quad & \iint_R f(x) \max_i \{u_i(x) - \lambda_i\} dA \quad s.t. \\ & \mathbf{q}^T \boldsymbol{\lambda} = 0. \end{aligned} \quad (7)$$

It remains to show that the optimal partition $\{R_1^*, \dots, R_n^*\}$ for the original problem (1) can be recovered from problem (7). Let $\{I_1^*(\cdot), \dots, I_n^*(\cdot)\}$ denote the optimal solution to the LP relaxation (4) and consider any point $x \in R$ and the optimal solution $\boldsymbol{\lambda}^*$ to (7). Suppose \bar{i} is the index such that $u_{\bar{i}}(x) - \lambda_{\bar{i}}^*$ is maximal (assuming such an index is unique). From the complementary slackness conditions of problem (6), it must be the case that $I_{\bar{i}}^*(x) = 0$ for all indices i other than \bar{i} , and consequently that $I_{\bar{i}}^*(x) = 1$. This completes the proof. \square

Remark 2. The dual variables λ_i have a natural interpretation as *shadow prices* associated with the agents; specifically, suppose that a client at point x must pay a fee of $-(u_i(x) - \lambda_i)$ for selecting agent i . Obviously, each client will choose the agent for which his or her fee is minimized. Thus, the dual problem asks us to choose rates λ_i that maximize the overall revenue that the agents receive from the clients, subject to a cap on the total rate at which they are allowed to charge them.

3.2 Sub-regions for (2)

In this section we consider the structure of problem (2). Again, we provide a proof sketch here and refer the reader to Section A.2 of the online supplement for a rigorous proof:

Theorem 3. *Provided that either $u_i(x) > 0$ for all i and x or $u_i(x) < 0$ for all i and x (except for possibly a set of measure zero), the boundaries between any optimal sub-regions R_i^* and R_j^* to problem (2) are of the form*

$$\partial(R_i^*) \cap \partial(R_j^*) \subseteq \left\{ x \in R : \frac{u_i(x)}{u_j(x)} = \frac{\lambda_j^*}{\lambda_i^*} \right\}$$

where λ_i^* and λ_j^* are the optimal solutions to the dual problem

$$\begin{aligned} \underset{\boldsymbol{\lambda}}{\text{minimize}} \quad & \iint_R f(x) \max_i \{\lambda_i u_i(x)\} dA \quad s.t. \\ & \sum_{i=1}^n \lambda_i = 1 \\ & \lambda_i \geq 0 \quad \forall i. \end{aligned}$$

Moreover, if x and i are a point and an index such that $\lambda_i^* u_i(x) > \lambda_j^* u_j(x)$ for all $j \neq i$, then $x \in R_i^*$. Thus, the optimal partition $\{R_1^*, \dots, R_n^*\}$ can be recovered from the optimal dual variables $\lambda_1^*, \dots, \lambda_n^*$, and at optimality, it turns out that all sub-regions have the exact same utility $\iint_{R_i^*} f(x) u_i(x) dA$.

Proof sketch. We can similarly formulate problem (2) as an infinite-dimensional integer program given by

$$\begin{aligned} \underset{t, I_1(\cdot), \dots, I_n(\cdot)}{\text{maximize}} \quad & t \quad s.t. \\ & t \leq \iint_R f(x) u_i(x) I_i(x) dA \quad \forall i \\ & \sum_{i=1}^n I_i(x) = 1 \quad \forall x \\ & I_i(x) \in \{0, 1\} \quad \forall i, x. \end{aligned} \quad (8)$$

The discretization of the linear programming relaxation of (8) is given by

$$\begin{aligned}
& \underset{t, Z}{\text{maximize}} && t && s.t. \\
& t &\leq & \sum_{j=1}^N \epsilon f_j u_{ij} z_{ij} && \forall i \\
& \sum_{i=1}^n z_{ij} &= & 1 && \forall j \\
& z_{ij} &\geq & 0 && \forall i, j.
\end{aligned} \tag{9}$$

The dual problem to (9), which has variables $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\boldsymbol{\varsigma} \in \mathbb{R}^N$, is

$$\begin{aligned}
& \underset{\boldsymbol{\mu}, \boldsymbol{\varsigma}}{\text{minimize}} && \sum_{j=1}^N \varsigma_j && s.t. \\
& \epsilon f_j u_{ij} \mu_i + \varsigma_j &\geq & 0 && \forall i, j \\
& -\sum_{i=1}^n \mu_i &= & 1 \\
& \mu_i &\leq & 0 && \forall i, j.
\end{aligned}$$

Introducing new variables $\lambda_i := -\mu_i$ and $\sigma_j := \varsigma_j / (\epsilon f_j)$, we can rewrite the above as

$$\begin{aligned}
& \underset{\boldsymbol{\lambda}, \boldsymbol{\sigma}}{\text{minimize}} && \sum_{j=1}^N \epsilon f_j \sigma_j && s.t. \\
& \sigma_j &\geq & u_{ij} \lambda_i && \forall i, j \\
& \sum_{i=1}^n \lambda_i &= & 1 \\
& \lambda_i &\geq & 0 && \forall i, j
\end{aligned}$$

which is a discretization of the problem

$$\begin{aligned}
& \underset{\boldsymbol{\lambda}, \sigma(\cdot)}{\text{minimize}} && \iint_R f(x) \sigma(x) dA && s.t. \\
& \sigma(x) &\geq & \lambda_i u_i(x) && \forall i, x \\
& \sum_{i=1}^n \lambda_i &= & 1 \\
& \lambda_i &\geq & 0 && \forall i
\end{aligned}$$

which is equivalent to the convex problem

$$\begin{aligned}
& \underset{\boldsymbol{\lambda}}{\text{minimize}} && \iint_R f(x) \max_i \{\lambda_i u_i(x)\} dA && s.t. \\
& \sum_{i=1}^n \lambda_i &= & 1 \\
& \lambda_i &\geq & 0 && \forall i.
\end{aligned} \tag{10}$$

As before, let $\{I_1^*(\cdot), \dots, I_n^*(\cdot)\}$ denote the optimal solution to the LP relaxation of (8) and consider any point $x \in R$ and the optimal solution $\boldsymbol{\lambda}^*$ to (10). Again, if the index \bar{i} that maximizes $\lambda_i u_i(x)$ is unique, it must be the case that $I_{\bar{i}}^*(x) = 1$ and $I_i^*(x) = 0$ for all other i . In addition, by our initial assumption that $u_i(x) > 0$ for all i and x or $u_i(x) < 0$ for all i and x (except for possibly a set of measure zero), it is not hard to show (by a simple perturbation argument) that we must have $\lambda_i^* > 0$ for all i . Complementary slackness then tells us that $t^* = \iint_R f(x) u_i(x) I_i^*(x) dA$ for all i at optimality, i.e. the sub-regions all have the exact same utility. \square

Remark 4. The dual variables λ_i again have a natural interpretation as shadow prices associated with the agents; specifically, suppose that a client at point x must pay a fee of $-\lambda_i u_i(x)$ for selecting agent i . Obviously, each client will choose the agent for which his or her fee is minimized. Again, the dual problem asks us to choose rates λ_i that maximize the overall revenue that the agents receive from the clients, subject to a cap on the total rate at which they are allowed to charge them.

4 Solving (1) and (2)

In this section we show that problems (1) and (2) can be solved efficiently using convex optimization. Specifically, we show that *subgradients* [8] to the dual problems (7) and (10) are cheap to compute, and therefore the optimal partition can be computed quickly using, for example, an *analytic center cutting plane method* [7].

4.1 Computing subgradients for (1)

It is straightforward to verify that the vector $\mathbf{g} \in \mathbb{R}^n$, defined by setting

$$g_i := - \iint_{R_i} f(x) dA,$$

is a subgradient for the objective function

$$h(\boldsymbol{\lambda}) := \iint_R f(x) \max_i \{u_i(x) - \lambda_i\} dA$$

for the dual problem (7). To see this, consider two vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ and the corresponding partitions $\{R_1, \dots, R_n\}$ and $\{R'_1, \dots, R'_n\}$. We want to show that $h(\boldsymbol{\lambda}') \geq h(\boldsymbol{\lambda}) + \mathbf{g}^T(\boldsymbol{\lambda}' - \boldsymbol{\lambda})$, i.e. that

$$\iint_R f(x) \max_i \{u_i(x) - \lambda'_i\} dA \geq \iint_R f(x) \max_i \{u_i(x) - \lambda_i\} dA + \mathbf{g}^T(\boldsymbol{\lambda}' - \boldsymbol{\lambda})$$

or equivalently that

$$\iint_R f(x) \max_i \{u_i(x) - \lambda'_i\} dA \geq \sum_{i=1}^n \iint_{R_i} f(x)(u_i(x) - \lambda_i) dA + g_i(\lambda'_i - \lambda_i).$$

Consider the right-hand side of the above; for each i , we have

$$\begin{aligned} \iint_{R_i} f(x)(u_i(x) - \lambda_i) dA + g_i(\lambda'_i - \lambda_i) &= \iint_{R_i} f(x)(u_i(x) - \lambda_i) dA - (\lambda'_i - \lambda_i) \iint_{R_i} f(x) dA \\ &= \iint_{R_i} f(x)(u_i(x) - \lambda'_i) dA \end{aligned}$$

and therefore we have

$$h(\boldsymbol{\lambda}') = \underbrace{\iint_R f(x) \max_i \{u_i(x) - \lambda'_i\} dA}_{(*)} \geq \sum_{i=1}^n \iint_{R_i} f(x)(u_i(x) - \lambda'_i) dA = h(\boldsymbol{\lambda}) + \mathbf{g}^T(\boldsymbol{\lambda}' - \boldsymbol{\lambda})$$

as desired, where $(*)$ follows from the fact that, by construction, the sub-regions of the partition $\{R'_1, \dots, R'_n\}$ are defined by taking the maximal value of $u_i(x) - \lambda'_i$ and are therefore maximal over all partitions. This completes the proof.

4.2 Computing subgradients for (2)

It is straightforward to verify that the vector $\mathbf{g} \in \mathbb{R}^n$, defined by setting

$$g_i := \iint_{R_i} f(x) u_i(x) dA,$$

is a subgradient for the objective function

$$h(\boldsymbol{\lambda}) := \iint_R f(x) \max_i \{\lambda_i u_i(x)\} dA$$

for the dual problem (10). To see this, consider two vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ and the corresponding induced partitions $\{R_1, \dots, R_n\}$ and $\{R'_1, \dots, R'_n\}$. We want to show that $h(\boldsymbol{\lambda}') \geq h(\boldsymbol{\lambda}) + \mathbf{g}^T(\boldsymbol{\lambda}' - \boldsymbol{\lambda})$, i.e. that

$$\iint_R f(x) \max_i \{\lambda'_i u_i(x)\} dA \geq \iint_R f(x) \max_i \{\lambda_i u_i(x)\} dA + \mathbf{g}^T(\boldsymbol{\lambda}' - \boldsymbol{\lambda})$$

or equivalently that

$$\iint_R f(x) \max_i \{\lambda'_i u_i(x)\} dA \geq \sum_{i=1}^n \iint_{R_i} \lambda_i f(x) u_i(x) dA + g_i(\lambda'_i - \lambda_i).$$

Consider the right-hand side of the above; for each i , we have

$$\begin{aligned} \iint_{R_i} \lambda_i f(x) u_i(x) dA + g_i(\lambda'_i - \lambda_i) &= \iint_{R_i} \lambda_i f(x) u_i(x) dA + (\lambda'_i - \lambda_i) \iint_{R_i} f(x) u_i(x) dA \\ &= \iint_{R_i} \lambda'_i f(x) u_i(x) dA \end{aligned}$$

and therefore we have

$$h(\boldsymbol{\lambda}') = \underbrace{\iint_R f(x) \max_i \{\lambda'_i u_i(x)\} dA}_{(*)} \geq \sum_{i=1}^n \iint_{R_i} \lambda'_i f(x) u_i(x) dA = h(\boldsymbol{\lambda}) + \mathbf{g}^T(\boldsymbol{\lambda}' - \boldsymbol{\lambda})$$

as desired, where $(*)$ follows from the fact that, by construction, the sub-regions of the partition $\{R'_1, \dots, R'_n\}$ are defined by taking the maximal value of $\lambda_i u_i(x)$ and are therefore maximal over all partitions. This completes the proof.

4.3 Algorithms for (1) and (2)

For the sake of completeness, we give formal descriptions of two algorithms, **MaxSumPartition** and **MaxMinPartition**, that solve problems (1) and (2), in Algorithms 1 and 2.

Remark 5. We would like to point out two advantages that Algorithms 1 and 2 possess over existing methods. First, we compute $\boldsymbol{\lambda}^*$ by solving a *convex* optimization problem for which subgradients are cheap to compute. Therefore, these approaches inherit better theoretical convergence properties than, say, the scheme of [33, 34], which uses a gradient descent method for the case where $u_i(x) = -\|x - p_i\|^2$ and whose associated objective function (i.e. the equivalent of $h(\boldsymbol{\lambda})$) is not convex (although the scheme proposed therein is proven to be globally convergent). Secondly, the scheme of [33, 34] requires an explicit expression for the boundary components of the power cells R_i (which are line segments), while ours merely requires that we be able to integrate over the R_i , say using Monte Carlo or quasi-Monte Carlo approximation. This is not an issue when $R \subset \mathbb{R}^2$ (since explicitly representing the boundary components – curves or line segments – is not difficult), but when $R \subset \mathbb{R}^d$ for high d , these boundary components may be difficult to enumerate. Monte Carlo and quasi-Monte Carlo integration methods, on the other hand, do not suffer from this “curse of dimensionality” because they involve merely sampling a large collection of points in R and estimating integrals by working with those points that lie in each region R_i ; the convergence rate of the integral is inversely proportional to the square root of the number of samples \sqrt{N} . Of course, such methods bring with them their own drawbacks, as described in Section 9.9.6 of [37]:

Such a convergence rate *does not* depend on the dimension n of the integration domain, and this is a most relevant feature of the Monte Carlo method. However, it is worth noting that the convergence rate is independent of the *regularity* of f ; thus, unlike interpolatory quadratures, Monte Carlo methods *do not* yield more accurate results when dealing with smooth integrands.

[The convergence rate of $1/\sqrt{N}$] is extremely weak and in practice one does often obtain poorly accurate results. A more efficient implementation of Monte Carlo methods is based on composite approach or semi-analytical methods; an example of these techniques is provided in [28], where a composite Monte Carlo method is employed for the computation of integrals over hypercubes in \mathbb{R}^n .

Input: A connected, polygonal region R with non-empty interior, a probability density function $f(\cdot)$ defined on R , a collection of n utility density functions $u_i(\cdot)$, a vector $\mathbf{q} \in \mathbb{R}_+^n$ such that $\sum_i q_i = 1$, and a threshold ϵ .

Output: A partition of R into n regions R_1, \dots, R_n that solves problem (1) within tolerance ϵ .

Note: this is simply a standard analytic center cutting plane method applied to problem (7).

Define the initial polyhedron by $\Lambda = \{\lambda \in \mathbb{R}^n : \mathbf{q}^T \lambda = 0 \text{ and } \|\lambda\|_\infty \leq M\}$ for a threshold M ;

/ See Lemma 14 of the online supplement to see how to construct a suitable M . */*

while $\text{vol}(\Lambda) > \epsilon$ **do**

 Let λ^0 be the analytic center of Λ ;

for $i \in \{1, \dots, n\}$ **do**

 Let R_i denote the sub-region in R for which $u_i(x) - \lambda_i^0$ is strictly maximal;

end

 Allocate the remaining mass of R (i.e. that which has not been assigned to a subset R_i , if any) lexicographically;

/ This lexicographic allocation will not generally be feasible for the original partitioning problem. */*

for $i \in \{1, \dots, n\}$ **do**

 Set $g_i := -\iint_{R_i} f(x) dA$;

end

 Set $\Lambda := \Lambda \cap \{\lambda : \mathbf{g}^T \lambda \geq \mathbf{g}^T \lambda^0\}$;

end

Let λ^0 be the analytic center of Λ ;

for $i \in \{1, \dots, n\}$ **do**

 Let R_i denote the sub-region in R for which $u_i(x) - \lambda_i^0$ is strictly maximal;

end

Allocate the remaining mass of R (i.e. that which has not been assigned to a subset R_i , if any) as described in Section B.2 of the online supplement;

return $\{R_1, \dots, R_n\}$;

Algorithm 1: Algorithm MaxSumPartition partitions a given region into sub-regions with pre-specified masses while maximizing the overall utility of n agents.

Given a sample $x \in R$, we can easily determine the region R_i containing x by merely checking which index i maximizes either $u_i(x) - \lambda_i$ or $\lambda_i u_i(x)$ (depending on what problem we are solving). Of course, we will eventually desire some kind of expression for the boundary components of the optimal regions R_1^*, \dots, R_n^* , be it implicit or explicit.

5 Applications revisited

Here we discuss the implications of Theorems 1 and 3 on the applications listed in Section 2.

Power diagrams By applying Theorem 1 to the case where $u_i(x) = -\|x - p_i\|^2$ for points $p_i \in R$, we find concise proofs of Theorems 1 and 3 and Corollary 1 from [5] (and equivalently Theorem 4.1 from [33] and Theorem 3.1 from [34]), all of which say (in one form or another) that the boundaries between optimal sub-regions to problem (1) are a power diagram, and therefore for any set of points $\{p_1, \dots, p_n\} \subset R$ and any constraint vector \mathbf{q} with $q_i \geq 0$ for all i and $\sum_i q_i = 1$, there exists a power diagram based at the points p_i such that $\iint_{C_i} f(x) dA = q_i$ for each cell C_i . As Algorithm 1 shows, it is simple to compute the optimal weight vector λ^* via the formulation (7), which improves over the control policy proposed in [33, 34] (which also provably converges to λ^*) as mentioned in Remark 5. Note that, at optimality, it may be the case that $p_i \notin R_i^*$ (since the cells in a power diagram are not guaranteed to contain their associated point).

An additional observation is that the boundaries between optimal sub-regions to problem (2) are level sets of the function $\|x - p_i\|^2 / \|x - p_j\|^2$ which are always arcs of a circle (specifically, an ‘‘Apollonian circle’’ [31]), as shown in Figure 1.

Fair division Consider the fair division problem (2) where each utility function $u_i(x)$ is a bivariate normal distribution with mean μ_i and covariance matrix Σ_i . Taking logarithms, it is easy to see that the boundaries between optimal sub-regions are level sets of the function

$$(x - \mu_i)^T \Sigma_i^{-1} (x - \mu_i) - (x - \mu_j)^T \Sigma_j^{-1} (x - \mu_j), \quad (11)$$

which are conic sections, as shown in Figure 2. It is worth mentioning that the sub-regions may not be connected.

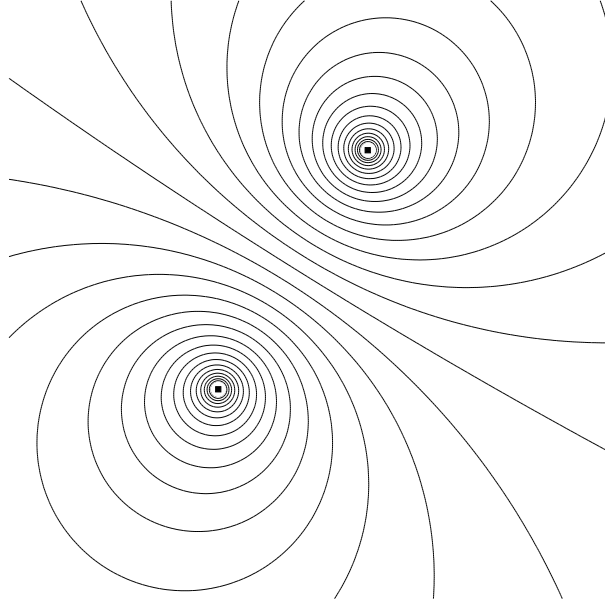


Figure 1: Apollonian circles, that is, the level sets of the function $\|x - p_i\|^2 / \|x - p_j\|^2$.

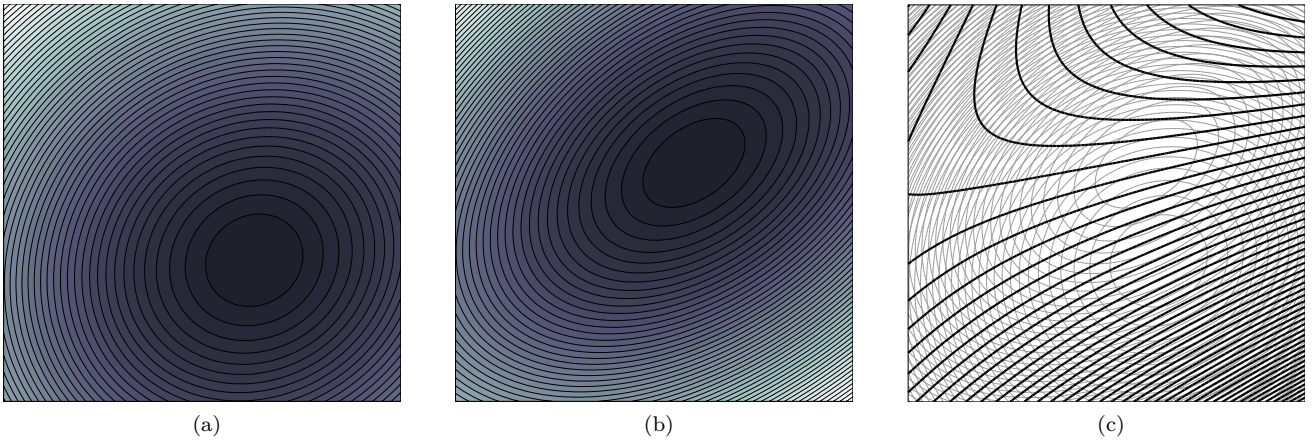


Figure 2: In (2a) and (2b) we have two normal distributions and in (2c) we show the level sets of the function (11); the light gray curves are the level sets of the normal distributions.

Input: A connected, polygonal region R with non-empty interior, a probability density function $f(\cdot)$ defined on R , a collection of n utility density functions $u_i(\cdot)$, and a threshold ϵ .

Output: A partition of R into n regions R_1, \dots, R_n that solves problem (2) within tolerance ϵ .

Note: this is simply a standard analytic center cutting plane method applied to problem (10).

Define the initial polyhedron by $\Lambda = \{\lambda \in \mathbb{R}^n : \sum_i \lambda_i = 1 \text{ and } \lambda \geq \mathbf{0}\}$;

while $\text{vol}(\Lambda) > \epsilon$ **do**

 Let λ^0 be the analytic center of Λ ;

for $i \in \{1, \dots, n\}$ **do**

 Let R_i denote the sub-region in R for which $\lambda_i^0 u_i(x)$ is strictly maximal;

end

 Allocate the remaining mass of R (i.e. that which has not been assigned to a subset R_i , if any) lexicographically;

for $i \in \{1, \dots, n\}$ **do**

 Set $g_i := \iint_{R_i} f(x) u_i(x) dA$;

end

 Set $\Lambda := \Lambda \cap \{\lambda : \mathbf{g}^T \lambda \geq \mathbf{g}^T \lambda^0\}$;

end

Let λ^0 be the analytic center of Λ ;

for $i \in \{1, \dots, n\}$ **do**

 Let R_i denote the sub-region in R for which $\lambda_i^0 u_i(x)$ is strictly maximal;

end

Allocate the remaining mass of R (i.e. that which has not been assigned to a subset R_i , if any) as described in Section B.2 of the online supplement;

return $\{R_1, \dots, R_n\}$;

Algorithm 2: Algorithm MaxMinPartition partitions a given region into sub-regions while maximizing the minimum utility of n agents.

Maximizing influence Theorem 1 tells us that when $u_i(x) = 1/\|x - p_i\|^2$, the boundaries between optimal sub-regions to problem (1) are quartic curves that are level sets of the function

$$\frac{1}{\|x - p_i\|^2} - \frac{1}{\|x - p_j\|^2}.$$

We are not aware of any name given to such curves, although they can easily be parameterized by expressing them in bipolar coordinate form and then applying a Euclidean transformation [25]. Such curves are shown in Figure 3. When we consider problem (2), it is easy to see that the boundaries between optimal sub-regions are Apollonian circles again.

Hospital districting When we consider problem (2) with $u_i(x) = \exp(-\|x - p_i\|)$, it is easy to see that the boundaries between optimal sub-regions are simply hyperbolas, that is, the level sets of

$$\|x - p_i\| - \|x - p_j\|.$$

Facility districting In the paper [12] we have previously considered problems (1) and (2) with $u_i(x) = -\|x - p_i\|$; it is easy to see that the boundaries between optimal sub-regions to (1) are hyperbolas and that the boundaries between optimal sub-regions to (2) are Apollonian circles.

As mentioned previously, when the region R contains obstacles (or “holes”), it may instead be appropriate to model the workload in a region R_i as

$$\iint_{R_i} f(x) d(x, p_i) dA,$$

where $d(x, p_i)$ is the length of the shortest path between x and p_i . We then find that the optimal sub-regions to (1) are *piecewise* hyperbolic arcs, as shown in Figure 4. It turns out that the sub-regions are *relatively star convex* to R and the points p_i : that is, if point x is contained in sub-region R_i , then the entire shortest path from x to p_i is contained in R_i as well; see Figure 5. In this sense, the sub-regions are both compact and connected. Given an optimal dual vector λ^* , we can construct the sub-regions R_i using the *continuous Dijkstra paradigm* in $\mathcal{O}(N^{5/3})$ steps (see [27]).

Districts for vehicle routing As we have seen in the preceding examples, in the vehicle districting problem as formulated previously, we are guaranteed to have either hyperbolic arcs (when $u_i(x) = -\|x - p_i\|$) or straight lines (when $u_i(x) = -\|x - p_i\|^2$) as boundary components, and in either case the optimal sub-regions are guaranteed to be

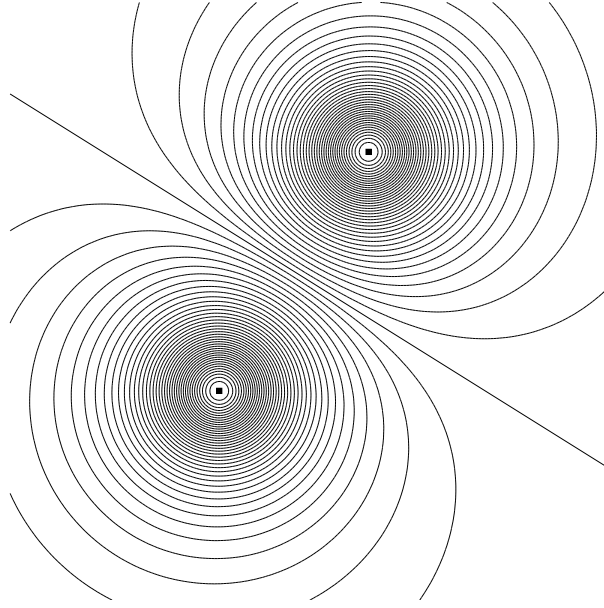


Figure 3: Level sets of the function $1/\|x - p_i\|^2 - 1/\|x - p_j\|^2$.

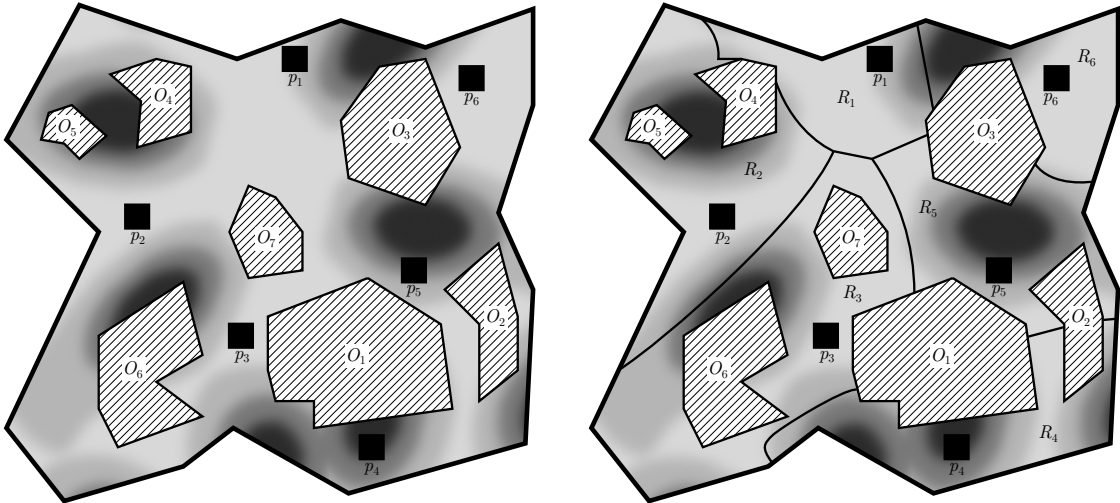


Figure 4: The partition above has hyperbolic arcs defining the boundary components between sub-regions.

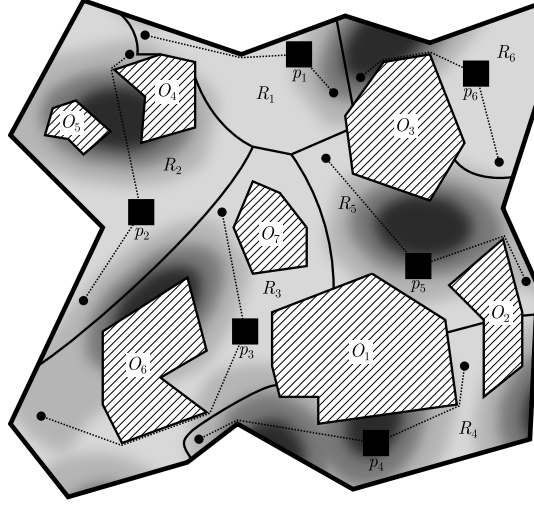


Figure 5: The partition from Figure 5 is relatively star-convex to R .

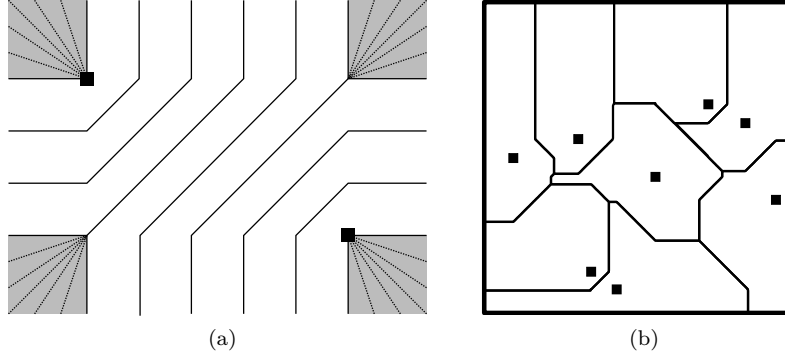


Figure 6: In (6a) we show the level sets of the function $\|x - p_i\|_1 - \|x - p_j\|_1$ and in (6b) we show an optimal solution to (1) on the unit square where $f(\cdot)$ is the uniform distribution, $q_i = 1/n$ for all n , and $u_i(x) = -\|x - p_i\|_1$. The problem of recovering an optimal partition given the dual variables λ_i^* is not entirely trivial because the boundary components may not be one-dimensional (i.e. the shaded regions in (6a)), although a partition satisfying the necessary properties can indeed be recovered as explained in Section B.1 of the online supplement.

at least connected (if not convex). If we insist that our boundary components be line segments (for the appearance of simplicity, for example) but we also want $p_i \in R_i^*$ for all i (which does not necessarily hold for a power diagram), another possibility is to use $u_i(x) = -\|x - p_i\|_1$ or $u_i(x) = -\|x - p_i\|_\infty$; the resulting boundary components are shown in Figure 6.

Police dragnet design When $u_i(x) = \max\{a - b \log \|x - p_i\|, 0\}$, it is easy to verify that the boundaries between optimal sub-regions to problem (1) are Apollonian circles, by the same arguments put forth earlier in this section.

6 Economic applications

Our analysis of problems (1) and (2) can actually be applied to non-geographic contexts. In this section, we apply Theorems 1 and 3 to look at the interaction between an *aggregate demand system* and an *address model*, which are commonly encountered in discrete choice theory in economics. Section 6.1 combines the results of this paper with the theory developed in [2] to show how to efficiently compute a market-clearing price vector in a variety of demand models for a differentiated product. Section 6.2 describes a variant on the classical *Fisher exchange market* [15] in which consumers are distributed according to a continuum.

6.1 Computing a market-clearing price vector in an aggregate demand system

Let $\mathbf{p} = (p_1, \dots, p_n)$ denote the prices of n variants of a differentiated product. An *aggregate demand system* (ADS) $\mathbf{D} : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ is a vector-valued function $\mathbf{D}(\mathbf{p})$ such that $D_i(\mathbf{p})$ represents the total demand for variant i from a given population of consumers when the variants are priced according to \mathbf{p} . The paper [2] considers ADSs $\mathbf{D}(\cdot)$ satisfying four conditions:

(A1) $\mathbf{D}(\cdot)$ obeys the *gross substitutes property*:

$$\frac{\partial D_i}{\partial p_j} > 0 \quad \forall i \neq j.$$

(A2) $\mathbf{D}(\cdot)$ is invariant under scalar addition:

$$\mathbf{D}(\mathbf{p} + c) = \mathbf{D}(\mathbf{p}) \quad \forall c \in \mathbb{R}_+.$$

(A3) Aggregate demand for the product is constant:

$$\sum_{i=1}^n D_i(\mathbf{p}) = 1 \quad \forall \mathbf{p} \in \mathbb{R}_+^n.$$

(A4) A technical constraint on the partial derivatives of $\mathbf{D}(\cdot)$:

$$\varphi(p_1 - p_n, \dots, p_{n-1} - p_n) := \frac{\partial^{n-1} D_i}{\partial p_1 \dots [\partial p_i] \dots \partial p_n} > 0$$

where the right hand side is the $(n-1)$ -th partial derivative of $D_i(\cdot)$ with respect to all prices except for p_i .

As explained in that paper, conditions (A1)-(A4) are satisfied under many standard discrete choice models, such as the *logit*, *probit*, *linear probability*, and *CES* models. One can also relax condition (A3) by regarding the *fraction* of demand attributed to each of the n variants rather than the true aggregate demand (this is explained in the “Conclusions” section thereof).

An alternative model to the ADS is the *address model*, defined as follows:

(B1) Each of the n product variants is represented as a point x_i in a “characteristics space” \mathbb{R}^m .

(B2) There is a continuum of consumers distributed in \mathbb{R}^m according to a continuous and strictly positive density function $f(x)$, with

$$\iiint_{\mathbb{R}^m} f(x) dV = 1,$$

where “ dV ” denotes a volume differential in \mathbb{R}^m .

(B3) Each consumer purchases one unit of the variant that offers the greatest utility. The utility of a consumer located at x , purchasing variant i , is

$$u_i(x) = \alpha_i - \|x - x_i\|^2 - p_i$$

where α_i is a perceived “quality index” of variant i and p_i is the price of variant i as in the ADS.

Under the address model, we then see that the total demand for variant i , written $\tilde{D}_i(\mathbf{p})$, is

$$\tilde{D}_i(\mathbf{p}) = \iiint_{R_i} f(x) dV, \tag{12}$$

where R_i is the “market space” for variant i , i.e. the region for which variant i offers the greatest utility:

$$R_i = \{x \in \mathbb{R}^m : \alpha_i - \|x - x_i\|^2 - p_i \geq \alpha_j - \|x - x_j\|^2 - p_j \forall j \neq i\}.$$

Note that for any price vector \mathbf{p} , the partition R_1, \dots, R_n is simply a *power diagram* as described in Section 2. We will assume without loss of generality that $\alpha_i = 0$ for all i .

The major insight of [2] is that there exists an *equivalence* between the ADS and address models of demand:

Theorem (ADS-address equivalency). *Given any ADS $\mathbf{D}(\cdot)$ satisfying conditions (A1)-(A4), there exists a density function $f(\cdot)$ and a placement of points $\{x_1, \dots, x_n\} \subset \mathbb{R}^m$ (where it turns out that $m = n-1$) such that $\tilde{\mathbf{D}}(\mathbf{p}) = \mathbf{D}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}_+^n$, with $\tilde{\mathbf{D}}(\cdot)$ as defined in (12).*

It is also worth noting that the equivalence established above is *constructive*; the authors give a closed-form expression for the placement of points x_i and the consumer density function $f(\cdot)$ in terms of $\mathbf{D}(\cdot)$. We can apply our Theorem 1 to the above result to show that, given an ADS $\mathbf{D}(\cdot)$ and a vector $\mathbf{d} \in \mathbb{R}_+^n$ such that $\sum_i d_i = 1$, we can easily find a *market-clearing price vector*, that is, a vector \mathbf{p}^* such that $D_i(\mathbf{p}^*) = d_i$ for all i . Conceptually, we construct the placement of points $\{x_1, \dots, x_n\}$ and the consumer density function $f(\cdot)$ as in the above theorem, then solve an instance of problem (1) in which we set $R = \mathbb{R}^m$, $q_i = d_i$, and $u_i(x) = -\|x - x_i\|^2$. We find that the optimal solution $\boldsymbol{\lambda}^*$ to the dual problem (7) is precisely the desired price vector \mathbf{p}^* .

What is more striking, however, is that we can in fact solve problem (1) without taking any integrals whatsoever! Recall from Section 4.1 that the key to solving (1) is that we can easily construct a subgradient vector \mathbf{g} by defining

$$g_i := - \iiint_{R_i} f(x) dV.$$

However, by construction, we know that $\iiint_{R_i} f(x) dV = \tilde{D}_i(\mathbf{p}) = D_i(\mathbf{p})$, and thus the vector $-\mathbf{D}(\mathbf{p})$ is *itself* a subgradient vector for the dual problem (7). To conclude this section, Algorithm 3 describes formally how to find \mathbf{p}^* given the ADS $\mathbf{D}(\cdot)$.

Input: An ADS $\mathbf{D}(\cdot)$ that satisfies (A1)-(A4), a vector $\mathbf{d} \in \mathbb{R}_+^n$ such that $\sum_i d_i = 1$, and a threshold ϵ .

Output: A *market-clearing price vector*, i.e. a vector $\mathbf{p}^* \in \mathbb{R}_+^n$ such that $\|\mathbf{D}(\mathbf{p}^*) - \mathbf{d}\| \in \mathcal{O}(\epsilon)$.

Define the initial polyhedron by $\mathcal{P} = \{\mathbf{p} \in \mathbb{R}^n : \mathbf{d}^T \mathbf{p} = 0 \text{ and } \|\mathbf{p}\|_\infty \leq M \text{ for a threshold } M;$

/* See Lemma 14 of the online supplement to see how to construct a suitable M . */

while $\text{vol}(\mathcal{P}) > \epsilon$ **do**

 Let \mathbf{p}^0 be the analytic center of \mathcal{P} ;

 Set $\mathbf{g} := -\mathbf{D}(\mathbf{p}^0)$;

 Set $\mathcal{P} := \mathcal{P} \cap \{\mathbf{p} : \mathbf{g}^T \mathbf{p} \geq \mathbf{g}^T \mathbf{p}^0\}$;

end

Let \mathbf{p}^0 be the analytic center of \mathcal{P} ;

Set $\mathbf{p} := \mathbf{p}^0 - \min_i \{p_i^0\}$ (we do this so as to ensure that $\mathbf{p} \in \mathbb{R}_+^n$);

return \mathbf{p} ;

Algorithm 3: Algorithm MarketClearingADS finds a market-clearing price vector in an ADS.

6.2 Relating (1) and (2)

Using Theorems 1 and 3, we can show that the optimal solutions to problems (1) and (2) are related in a particular fashion:

Claim. Let R_1^*, \dots, R_n^* be an optimal solution to problem (2) with generic utility functions $\bar{u}_i(x)$ and define $q_i^* := \iint_{R_i^*} f(x) dA$ for all i . Then R_1^*, \dots, R_n^* is also an optimal solution to problem (1) with the same density $f(\cdot)$ and utility functions $u_i(x) = \log \bar{u}_i(x)$ with $q_i = q_i^*$.

Proof. This is immediate; R_1^*, \dots, R_n^* satisfies the KKT conditions for problem (1), as follows from our proofs of Theorems 1 and 3. \square

This result seems to be relevant in general problems of fair allocation of divisible goods [9], although we are not aware of its existence elsewhere in the literature. One particular application arises when we consider a continuous version of the classical *Fisher exchange market*, as discussed below.

Fisher's exchange market

Fisher's exchange market [15] is a special case of the general *Arrow-Debreu problem* [4] in which an economy consists of producers and consumers. A collection of N consumers have money to buy goods and maximize their (linear) utility functions; a collection of n producers sell their goods for money. We assume that each producer produces one unique good, so that the unit prices for the goods are denoted by a vector $\mathbf{p} \in \mathbb{R}_+^n$. Without loss of generality we assume that each producer has exactly one unit of his or her unique good to sell, which can be divided among the

consumers. Associated with each consumer is a utility vector $\mathbf{u}_i \in \mathbb{R}_+^n$, a budget b_i , and a decision vector $\mathbf{x}_i \in \mathbb{R}_+^n$. Each agent chooses a bundle of goods by solving the simple linear program

$$\begin{aligned} \underset{\mathbf{x}_i}{\text{maximize}} \quad & \mathbf{u}_i^T \mathbf{x}_i \quad s.t. \\ & \mathbf{p}^T \mathbf{x}_i \leq b_i \\ & \mathbf{x}_i \geq \mathbf{0}. \end{aligned} \quad (13)$$

The objective of a market organizer is to determine a price vector \mathbf{p}^* such that the market clears, i.e. that when each consumer selects his or her optimal decision vector \mathbf{x}_i^* , we have

$$\sum_{i=1}^N \mathbf{x}_i^* = \mathbf{e}$$

where $\mathbf{e} \in \mathbb{R}^n$ is a vector whose entries are all 1's. A classical result of Eisenberg and Gale [13] explains how to construct \mathbf{p}^* :

Theorem. *The optimal Lagrange multiplier for the equality constraints in the following optimization problem is a market-clearing price vector:*

$$\begin{aligned} \underset{\mathbf{x}_1, \dots, \mathbf{x}_N}{\text{maximize}} \quad & \sum_{i=1}^N b_i \log(\mathbf{u}_i^T \mathbf{x}_i) \quad s.t. \\ & \sum_{i=1}^N \mathbf{x}_i = \mathbf{e} \\ & \mathbf{x}_i \geq \mathbf{0} \quad \forall i. \end{aligned} \quad (14)$$

In keeping with the spirit of this paper, we will use a mild variant of Fisher's model in which agents are distributed according to a *continuum* $f(x)$ defined on a domain $R \in \mathbb{R}^m$, rather than by the index set $\{1, \dots, N\}$ (here \mathbb{R}^m would be a "characteristics space" in the language of Section 6.1). In this setting, an agent located at point $x \in R$ has a utility vector $\mathbf{u}(x) = (u_1(x), \dots, u_n(x))$, a given budget $b(x)$, and a decision vector $\mathbf{J}(x) = (J_1(x), \dots, J_n(x))$. The relevant equivalents of (13) and (14) are then

$$\begin{aligned} \underset{J_1(x), \dots, J_n(x)}{\text{maximize}} \quad & \sum_{i=1}^n u_i(x) J_i(x) \quad s.t. \\ & \sum_{i=1}^n p_i J_i(x) \leq b(x) \\ & J_i(x) \geq 0 \quad \forall i \end{aligned} \quad (15)$$

and

$$\begin{aligned} \underset{J_1(\cdot), \dots, J_n(\cdot)}{\text{maximize}} \quad & \iiint_R f(x) b(x) \log \left(\sum_{i=1}^n u_i(x) J_i(x) \right) dV \quad s.t. \\ & \iiint_R f(x) J_i(x) dV = 1 \quad \forall i \\ & J_i(x) \geq 0 \quad \forall i, x \end{aligned} \quad (16)$$

respectively.

Rather than clearing the market by selling fixed quantities of goods (which we would accomplish by solving (16)), we will show how to solve a related problem apropos of that encountered in Section 6.1: suppose that a market organizer wants to select a price vector \mathbf{p} with the intention of setting, for each good i , the *fraction of customers that prefer good i to all other goods*. The budgets of the agents, $b(x)$, and the amounts available of each good (which we assumed to be 1), are now disregarded. This is a useful model when the market consists of competing variants of products whose aggregate demand is *inelastic* (as opposed to the traditional Fisher market which is perfectly elastic), such as sanitation services or health insurance. For lack of a better phrase, we will call such a price vector an "allocation-clearing price vector".

Theorem 6. Let $\mathbf{q} \in \mathbb{R}_+^n$ be a vector such that $\sum_i q_i = 1$ representing desired consumer allocations and suppose that $f(x)$ is a probability density function such that $\iint_R f(x) dV = 1$ on a domain $R \subset \mathbb{R}^m$. Then the vector \mathbf{p} defined by setting $p_i = e^{\lambda_i^*}$ for all i , where λ^* is the optimal Lagrange multiplier for the top equality constraints in the following optimization problem, is an “allocation-clearing price vector”:

$$\begin{aligned} \underset{I_1(\cdot), \dots, I_n(\cdot)}{\text{maximize}} \quad & \iint_R f(x) \sum_{i=1}^n (\log u_i(x)) I_i(x) dV \quad \text{s.t.} \\ & \iint_R f(x) I_i(x) dV = q_i \quad \forall i \\ & \sum_{i=1}^n I_i(x) = 1 \quad \forall x \\ & I_i(x) \geq 0 \quad \forall i, x. \end{aligned} \quad (17)$$

Proof. Since (17) is an instance of (1), Theorem 1 says that at optimality, the region R_i^* (i.e. the region where $I_i^*(x) = 1$) consists of those points x where $\log u_i(x) - \lambda_i^* > \log u_j(x) - \lambda_j^*$ for all $j \neq i$, or equivalently, where $u_i(x)/e^{\lambda_i^*} = u_i(x)/p_i > u_j(x)/p_j$. Conversely, in order to maximize (15), a consumer located at point x will allocate its entire budget to precisely that same index i that maximizes $u_i(x)/p_i$. \square

For the sake of completeness, problem (18) below is simply the discrete analogue of problem (17), where we assume that $\mathbf{q} \in \mathbb{R}_+^N$ satisfies $\sum_i q_i = N$. We believe this to be of independent interest because Theorem 6 still holds, provided we allow consumers to be fractionally allocated to two or more goods if they are indifferent:

$$\begin{aligned} \underset{\mathbf{x}_1, \dots, \mathbf{x}_N}{\text{maximize}} \quad & \sum_{i=1}^N \log(\mathbf{u}_i)^T \mathbf{x}_i \quad \text{s.t.} \\ & \sum_{i=1}^N \mathbf{x}_i = \mathbf{q} \\ & \mathbf{e}^T \mathbf{x}_i = 1 \quad \forall i \\ & \mathbf{x}_i \geq 0 \quad \forall i. \end{aligned} \quad (18)$$

7 Computational complexity

Up to this point we have described how to solve the dual problems (7) and (10) efficiently by constructing subgradient vectors. However, we have not yet discussed the added computational complexity incurred by evaluating these integrals numerically. To this end, we find the following result (originally given in Section 7.4 of [20], but re-stated using the language of Section 9.9 of [37]) useful:

Theorem 7. Let $\Omega \subset \mathbb{R}^2$ be a domain of integration equipped with a triangulation \mathcal{T}_h consisting of N_T triangles, where h is the maximum edge length in \mathcal{T}_h . There exists a positive constant K_1 , independent of h , such that the error E induced using either the composite midpoint formula

$$\iint_{\Omega} f(x) dA \approx \sum_{T \in \mathcal{T}_h} \text{Area}(T) f(\text{centroid}(T))$$

or the composite trapezoidal formula

$$\iint_{\Omega} f(x) dA \approx \frac{1}{3} \sum_{T \in \mathcal{T}_h} \text{Area}(T) \sum_{j=1}^3 f(\text{vertex}_j(T))$$

is bounded by

$$|E| \leq K_1 h^2 \text{Area}(\Omega) M_2,$$

where M_2 is the maximum value of the modules of the second derivatives of the integrand $f(\cdot)$.

It follows that, if our desired error in integration is ϵ in our problem, then the maximum length of any edge in the triangulation must be at most $\epsilon^{1/2}(\text{Area}(R)M_2K_1)^{-1/2}$ (since we often have curved arcs separating the R_i , an exact triangulation is impossible, but the added computational complexity therein is beyond the scope of this paper). If we break R into N_T triangles, the maximum edge length will generally be $\mathcal{O}(\sqrt{\text{Area}(R)/N_T})$ and we therefore need to break R into $\mathcal{O}(\text{Area}(R)^2M_2/\epsilon)$ triangles. Thus, the complexity of evaluating the sub-gradient vectors in Section 4 is quadratic in $\text{Area}(R)$, linear in the maximum modules of the second derivatives of $f(\cdot)$, and inversely proportional to the desired precision ϵ .

8 Enforcing connectivity

We have seen previously that the optimal partition to an instance of (1) or (2) may not be connected. In practice, connectivity is clearly a desirable property; for example, by state constitution, statute, or guideline, 23 US states have passed rulings that their congressional districts must be contiguous [23]. Having observed that the optimal solution to problem (1) is always connected when $u_i(x) = -\|x - p_i\|$, one way to enforce connectivity of sub-regions is to augment problems (1) or (2) with a *penalty term* $-\|x - p_i\|$, giving the modified problems

$$\begin{aligned} \text{maximize}_{R_1, \dots, R_n} (1 - \mu) \sum_{i=1}^n \iint_{R_i} f(x) u_i(x) dA - \mu \sum_{i=1}^n \iint_{R_i} f(x) \|x - p_i\| dA \quad s.t. \end{aligned} \quad (19)$$

$$\begin{aligned} \iint_{R_i} f(x) dA &= q_i \quad \forall i \\ R_i \cap R_j &= \emptyset \quad \forall i \neq j \\ \bigcup_i R_i &= R \end{aligned}$$

and

$$\begin{aligned} \text{maximize}_{R_1, \dots, R_n} (1 - \mu) \min_i \left\{ \iint_{R_i} f(x) u_i(x) dA \right\} - \mu \sum_{i=1}^n \iint_{R_i} f(x) \|x - p_i\| dA \quad s.t. \end{aligned} \quad (20)$$

$$\begin{aligned} R_i \cap R_j &= \emptyset \quad \forall i \neq j \\ \bigcup_i R_i &= R. \end{aligned}$$

It is obvious that problem (19) is itself merely an instance of (1) (with a modified utility function) and thus its boundary components may be solved using duality as before. It is not hard to show that the dual to problem (20) is

$$\begin{aligned} \text{minimize}_{\lambda} \iint_R f(x) \max_i \{ \lambda_i u_i(x) - \mu \|x - p_i\| \} dA \quad s.t. \\ \sum_i \lambda_i &= 1 - \mu \\ \lambda_i &\geq 0 \quad \forall i \end{aligned}$$

and consequently the optimal boundaries to (20) must satisfy

$$\lambda_i u_i(x) - \mu \|x - p_i\| = \lambda_j u_j(x) - \mu \|x - p_j\|.$$

Figure 7 shows the effect of this penalty term on an instance of (20) where we use the gravity model utility functions given by $u_i(x) = 1/\|x - p_i\|^2$.

9 Dynamic partitioning

In this section we consider the problem of partitioning a region optimally when the density $f(x)$ varies over time. For ease of exposition we shall focus on the case where R is a planar region, although the analysis herein extends naturally to higher dimensions. We consider the case where $f(x; t)$ is a probability density on $R \in \mathbb{R}^2$ that evolves according to a *vector field* $\vec{V}(x, t) : R \times \mathbb{R} \rightarrow \mathbb{R}^2$ that maps a given point $x \in R$ at a specified time t to a vector

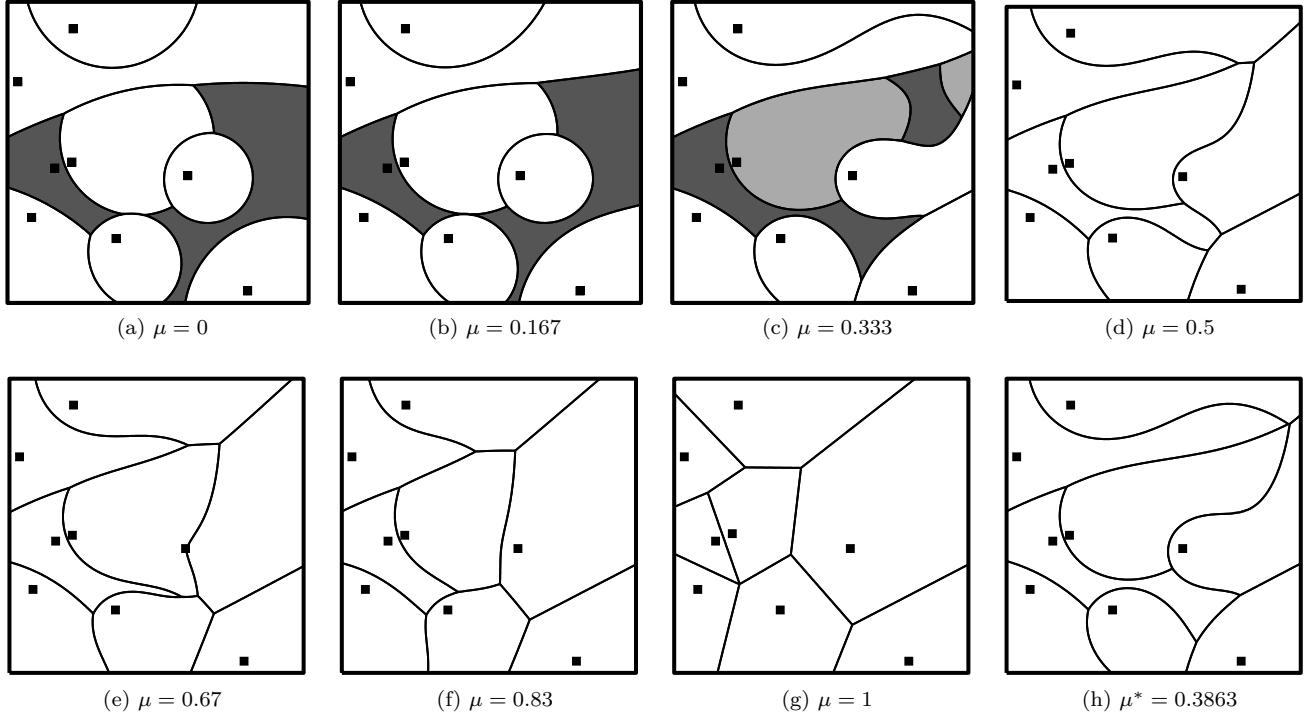


Figure 7: The optimal partitions to problem (20) with varying μ , where $f(x)$ is the uniform distribution and we use the gravity model utility functions given by $u_i(x) = 1/\|x - p_i\|^2$ (it turns out that we can parameterize the boundary curves efficiently by using a transformation to bipolar two-center coordinates). Disconnected regions are indicated by shading. Figure (7h) shows the partition induced when $\mu = \mu^* = 0.3863$ is the “threshold” value (which we found using the method of bisection) for which the partition is connected.

$(v_1(x, t), v_2(x, t))$; the physical interpretation of this system, naturally, is that an infinitesimal amount of mass located at point x moves in the direction $(v_1(x, t), v_2(x, t))$ at time t at a speed of $\|(v_1(x, t), v_2(x, t))\|$. Such a model of transport is canonical in a variety of contexts, including population migration [42], meteorology [17], and oceanography [29]. In such a setting, the density $f(x; t)$ is known to obey the *advection equation*

$$\frac{\partial f(x; t)}{\partial t} + \nabla \cdot (f(x; t) \vec{V}(x, t)) = 0 \quad (21)$$

where $\nabla \cdot$ is the *divergence operator*, defined as

$$\nabla \cdot \begin{pmatrix} h_1(x_1, x_2; t) \\ h_2(x_1, x_2; t) \end{pmatrix} := \frac{\partial h_1(x_1, x_2; t)}{\partial x_1} + \frac{\partial h_2(x_1, x_2; t)}{\partial x_2}.$$

In this section we will show how equation (21) allows us to define the changes in the optimal Lagrange multiplier vector λ^* over time, and thereby the optimal partition $R_1^*(t), \dots, R_n^*(t)$ (since λ^* uniquely defines the optimal sub-regions R_i^*). Let $\lambda^*(t)$ denote the optimal Lagrange multiplier vector at time t and let $R_i(\lambda^*(t))$ denote the i th sub-region constructed according to Theorem 1 (it is of course true that $R_i(\lambda^*(t)) = R_i^*(t)$, but for notational purposes we prefer to emphasize the dependence of $R_i^*(t)$ on $\lambda^*(t)$). Using the result of Section 4.1, we can easily see that the optimality conditions of problem (7) simply require that

$$F_i(\lambda; t) = q_i \quad (22)$$

for all $i \in \{1, \dots, n\}$ and all t , where we have defined (for ease of notation)

$$F_i(\lambda; t) := \iint_{R_i(\lambda)} f(x; t) dA.$$

Since $\sum_i q_i = \iint_R f(x; t) = 1$ for all t , we find that one of these constraints is redundant and therefore it will suffice to require that (22) holds for $i \in \{1, \dots, n-1\}$ and all t ; we also commensurately assign $\lambda_n = -(q_1 \lambda_1 + \dots + q_{n-1} \lambda_{n-1})/q_n$, to comply with the dual constraint that $\mathbf{q}^T \lambda = 0$. We shall now use (22) to define the partial derivatives $\partial \lambda_i^* / \partial t$. We first introduce two well-known lemmas:

Lemma 8. *Let R_τ be a family of compact regions in the plane defined by*

$$R_\tau := \{x \in \mathbb{R}^2 : h(x) \leq \tau\}$$

where $h(x) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth function. Suppose $f(x)$ is a density on \mathbb{R}^2 and define

$$m(\tau) := \iint_{R_\tau} f(x) dA$$

for all τ . Then

$$\frac{dm}{d\tau} = \int_{\partial R_\tau} \frac{f(x)}{\|\nabla h\|} ds,$$

where ds denotes a line integral over the boundary of R_τ , ∂R_τ .

Proof. This is a special case of the *coarea formula* [21]. □

Lemma 9. (Divergence theorem [1]) *Let R be a compact planar region with a smooth boundary. If $\vec{W}(x)$ is a smooth vector field defined on R , then*

$$\iint_R \nabla \cdot \vec{W} dA = \int_{\partial R} \vec{w} \cdot \vec{n} ds,$$

where \vec{n} denotes the outward-facing unit normal vector pointing out of R .

Note that Lemma 8 gives us a clean expression for the partial derivatives $\partial F_i / \partial \lambda_j$:

$$\frac{\partial F_i}{\partial \lambda_j} = \begin{cases} 0 & \text{if } \partial R_i \cap \partial R_j = \emptyset \\ \sum_{j' : \partial R_i \cap \partial R_{j'} \neq \emptyset} \int_{\partial R_i \cap \partial R_{j'}} \frac{f(x; t)}{\|\nabla u_i(x) - \nabla u_{j'}(x)\|} ds & \text{if } i = j \\ - \int_{\partial R_i \cap \partial R_j} \frac{f(x; t)}{\|\nabla u_i(x) - \nabla u_j(x)\|} ds & \text{otherwise} \end{cases}$$

where we have suppressed the dependency of the R_i 's on λ purely for notational compactness. Similarly, combining equation (21) with Lemma 9 gives us a clean expression for the partial derivatives $\partial F_i / \partial t$:

$$\begin{aligned}
\frac{\partial F_i}{\partial t} &= \frac{\partial}{\partial t} \iint_{R_i(\lambda)} f(x; t) dA \\
&= \iint_{R_i(\lambda)} \frac{\partial}{\partial t} f(x; t) dA \\
&= - \iint_{R_i(\lambda)} \nabla \cdot (f(x; t) \vec{V}(x, t)) dA \\
&= - \int_{\partial R_i(\lambda)} (f(x; t) \vec{V}(x, t)) \cdot \vec{n} ds.
\end{aligned}$$

We thus have in hand expressions for $\partial F_i / \partial \lambda_j$ and $\partial F_i / \partial t$. The following result allows us to describe the optimal dual variables λ^* in terms of t :

Theorem 10. (Special case of the implicit function theorem [1]) *Consider the system of $n - 1$ equations in n variables*

$$\begin{aligned}
F_1(\lambda_1, \dots, \lambda_{n-1}; t) - q_1 &= 0 \\
&\vdots \\
F_{n-1}(\lambda_1, \dots, \lambda_{n-1}; t) - q_{n-1} &= 0
\end{aligned}$$

and a point (λ_0, t_0) that satisfies the system. Suppose that each of the functions F_i has continuous first partial derivatives with respect to each of the variables λ_i and t near (λ_0, t_0) . Finally, suppose that

$$\det \left[\frac{\partial F_i}{\partial \lambda_j} \right] \Big|_{(\lambda_0, t_0)} \neq 0, \tag{23}$$

where $[\partial F_i / \partial \lambda_j]$ denotes the Jacobian matrix of functions F_1, \dots, F_{n-1} with respect to $\lambda_1, \dots, \lambda_{n-1}$. Then there exist functions $\phi_1(t), \dots, \phi_{n-1}(t)$ such that

$$\phi_i(t_0) = \lambda_i$$

for $i \in \{1, \dots, n - 1\}$ and such that the equations

$$\begin{aligned}
F_1(\phi_1(t), \dots, \phi_{n-1}(t); t) - q_1 &= 0 \\
&\vdots \\
F_{n-1}(\phi_1(t), \dots, \phi_{n-1}(t); t) - q_{n-1} &= 0
\end{aligned}$$

hold for all t sufficiently near t_0 . Moreover,

$$\frac{\partial \phi_j}{\partial t} = - \frac{\det \left[\frac{\partial F_i}{\partial \lambda_1 \dots \partial t \dots \partial \lambda_{n-1}} \right]}{\det \left[\frac{\partial F_i}{\partial \lambda_j} \right]},$$

where the numerator denotes the matrix obtained by replacing the j th column of $[\partial F_i / \partial \lambda_j]$ with the vector of partial derivatives $\partial F_i / \partial t$.

Corollary 11. *If $\lambda^*(t_0)$ is an optimal solution to problem (7) at time t_0 , and if $f(x; t)$ evolves according to equation (21), then the optimal dual variables $\lambda^*(t)$ satisfy the differential equation*

$$\left. \frac{\partial \lambda_i^*}{\partial t} \right|_{t=t_0} = - \frac{\det \left[\frac{\partial F_i}{\partial \lambda_1 \dots \partial t \dots \partial \lambda_{n-1}} \right]}{\det \left[\frac{\partial F_i}{\partial \lambda_j} \right]} \Big|_{t=t_0}. \tag{24}$$

Remark 12. We can further elaborate on a sufficient condition for (23) to hold. Let $\mathfrak{J} = [\mathfrak{J}_{ij}] := [\partial F_i / \partial \lambda_j]$ denote the Jacobian matrix of partial derivatives with respect to λ_1 through λ_{n-1} as before and note that $\mathfrak{J}_{ij} = 0$ if regions i and j do not share a boundary. Construct a graph G with $n - 1$ nodes, where nodes i and j share an edge if

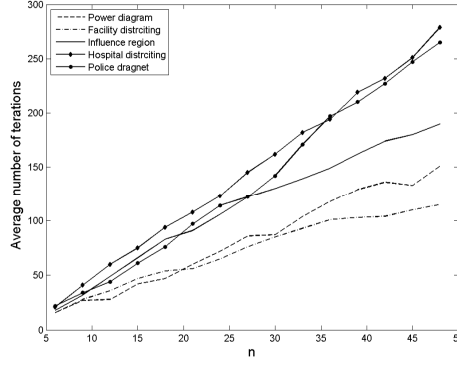


Figure 8: The average number of cutting plane method iterations for various utility functions.

regions i and j share a boundary (G may not be connected since region n does not have a corresponding node). Assume without loss of generality that \mathfrak{J} is decomposed into a block-diagonal form wherein each block \mathfrak{B}_k consists of the connected components of G . Further note that, by construction, \mathfrak{J} is diagonally dominant. If $\mathfrak{J}_{ij} < 0$ for all neighboring regions i and j (which always holds in all of our examples if $f(x) > 0$, for instance), then in each block, there exists at least one row (namely, any row i such that regions i and n share a boundary) at which this diagonal dominance is strict. Each block \mathfrak{B}_k is therefore *irreducibly diagonally dominant* and thus nonsingular [43], which guarantees nonsingularity of \mathfrak{J} .

10 Computational experiments

In what follows we give the results of three numerical simulations: in the first two simulations, we solve various instances of (1) and (2) when R is either the unit square or a geographic map, and $f(\cdot)$ is either the uniform distribution or a population density. In the second simulation we consider a dynamic partitioning problem among four agents in the unit square and show how Corollary 11 can be used effectively.

10.1 Static partitioning

In this section, we provide the results of two numerical simulations: in the first simulation, R is the unit square and $f(\cdot)$ is the uniform distribution; in the second, R is a map of Ramsey County, Minnesota, and $f(\cdot)$ is a population density. For all problems we use $q_i = 1/n$ for all i ; the agents' locations are randomly chosen. Figure 8 shows the number of cutting plane iterations (averaged over 5 random samples) required for $n = 6$ through $n = 50$ for the unit square. Figures 9 and 11 show the convergence of our algorithm for $n = 6$ in the unit square and $n = 7$ in Ramsey County from both a primal and dual perspective. Figures 10 and 12 show the various optimal partitions that are computed by our algorithm. For all simulations we used a tolerance threshold of 1%.

10.2 Dynamic partitioning

In this section, we consider a problem in which the density $f(\cdot)$ varies over time as in equation (21). We let R be the unit square and we let $f(x; t = 0)$ be a normal distribution with mean $(0.5, 0.5)$ and variance 0.3 with zero covariance. We let $u_i(x) := -\|x - p_i\|$ where the points p_i are regularly spaced in R .

As a vector field $\vec{V}(x; t)$ we use $v_1(x_1, x_2, t) := -\sin(x'_1) \cos(x'_2) + 0.1$ and $v_2(x_1, x_2, t) := \cos(x'_1) \sin(x'_2)$, where x'_1 and x'_2 are obtained by applying an affine map to x_1 and x_2 (we use such a map so as to have a vector field that is not aligned with the coordinate axes), and we also apply a dampening filter to force $\|\vec{V}(x, t)\|$ to be small when x is near the boundary of R (this allows us to sidestep the issue of mass entering or exiting R). The field is shown in Figure 13. Note that we have chosen to keep \vec{V} constant over time in order to make the flow of $f(x; t)$ more recognizable.

In order to simulate the advection over time we use the Clawpack simulator [22] for the period $t \in [0, 10]$, with 200×200 grid cells, using a timestep of $\Delta t = 0.01$. For each of the 1000 advection iterations we compute the optimal dual variables $\lambda_i^*(t)$ and the optimal partitions, which are shown in Figure 14. We also compute

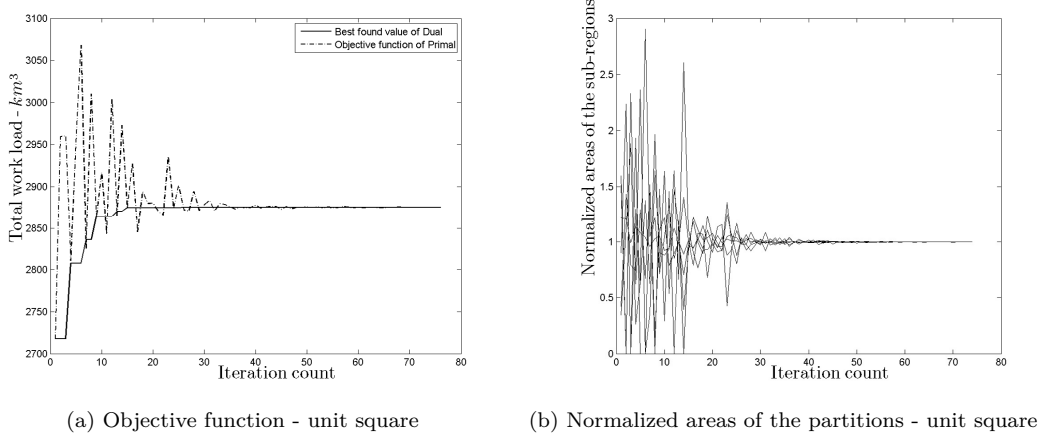


Figure 9: Convergence of the analytic center cutting plane algorithm for the unit square.

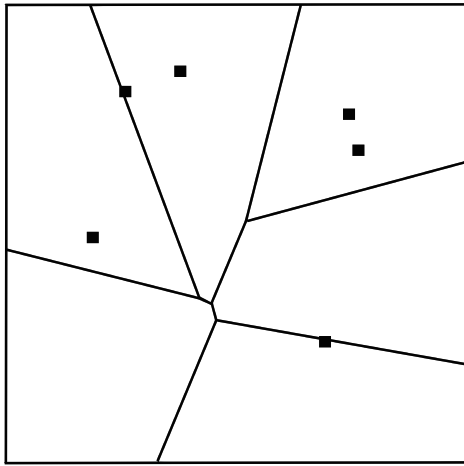
approximately optimal dual variables $\lambda_i^\dagger(t)$ using Corollary 11 as follows: for $t \in \{0, 1, \dots, 9\}$, we let $\lambda_i^\dagger(t) = \lambda_i^*(t)$, and for non-integer t , we let $\lambda_i^\dagger(t)$ be the value of $\lambda_i^*(t)$ as prescribed by equation (24). Thus, $\lambda_i^\dagger(t)$ is a piecewise linear function that “resets” itself whenever t is an integer. For purposes of comparison we also let $\lambda_i^\ddagger(t)$ be a *step* function that “resets” itself whenever t is an integer, that is, $\lambda_i^\ddagger(t) := \lambda_i^*(\lfloor t \rfloor)$. In summary, $\lambda_i^\dagger(t)$ is a first-order approximation of $\lambda_i^*(t)$ and $\lambda_i^\ddagger(t)$ is a zeroth-order approximation. Figure 15a shows the trajectories of these two approximations against the true optimal trajectories $\lambda_i^*(t)$. Figure 15b shows the resulting values of $F_i(\mathbf{\lambda}; t)$ under these two approximations; note that the regions defined by $\lambda_i^\dagger(t)$ are, not surprisingly, consistently more balanced than those of the step approximation $\lambda_i^\ddagger(t)$.

11 Acknowledgements

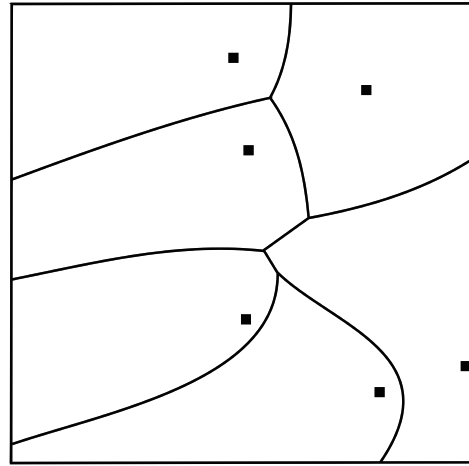
The authors thank Yinyu Ye and Shuzhong Zhang for their helpful suggestions.

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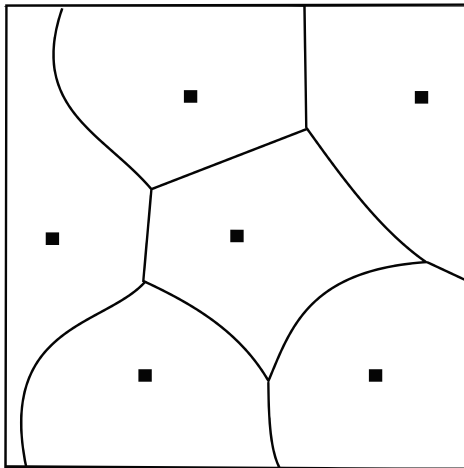
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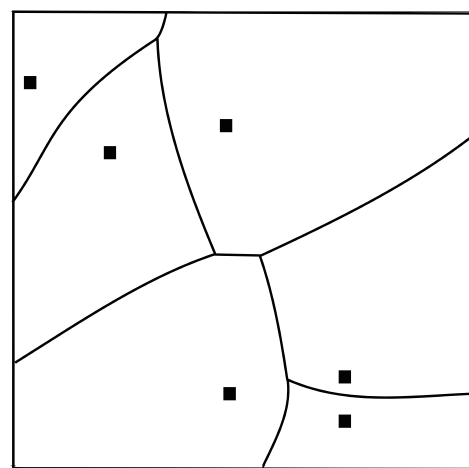
(a) Power Diagram



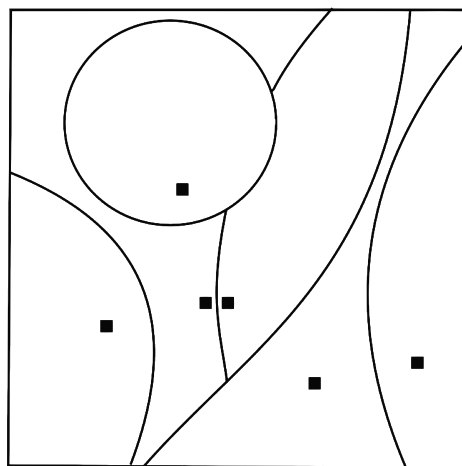
(b) Facility districting



(c) Influence regions



(d) Hospital districting



(e) Police dragnet design

Figure 10: Partitioning the unit square for various utility functions.

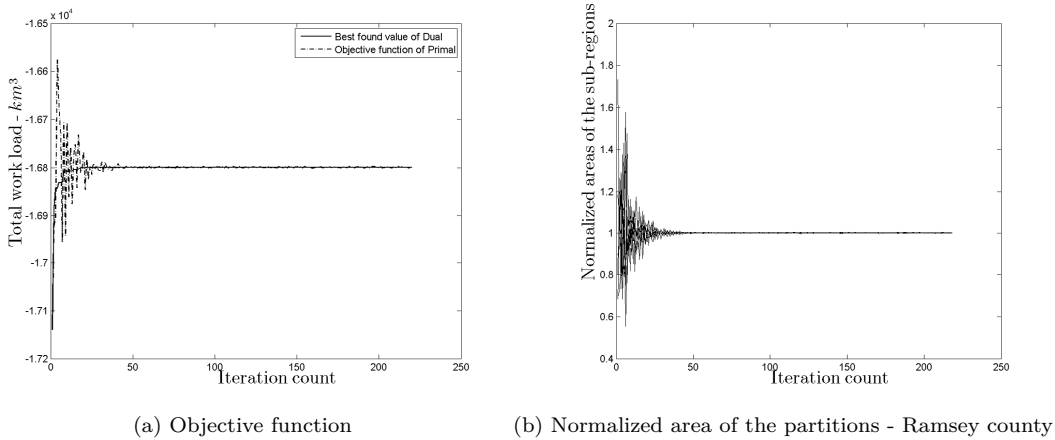
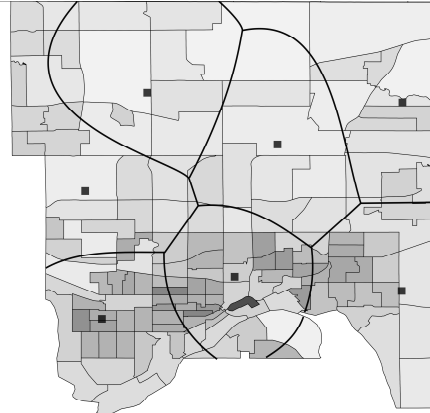


Figure 11: Performance of the analytic center cutting plane algorithm for Ramsey county.

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(a) Power Diagram



(b) Influence regions



(c) Hospital districting



(d) Police dragnet design



(e) Facility districting

Figure 12: Partitioning Ramsey County for various utility functions.

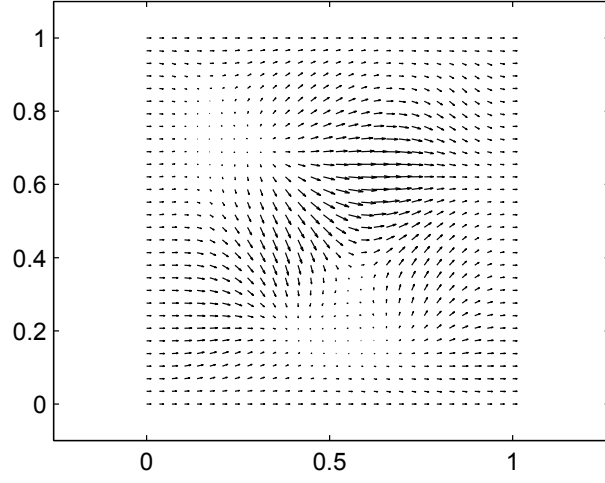


Figure 13: The vector field $\vec{V}(x_1, x_2; t)$.

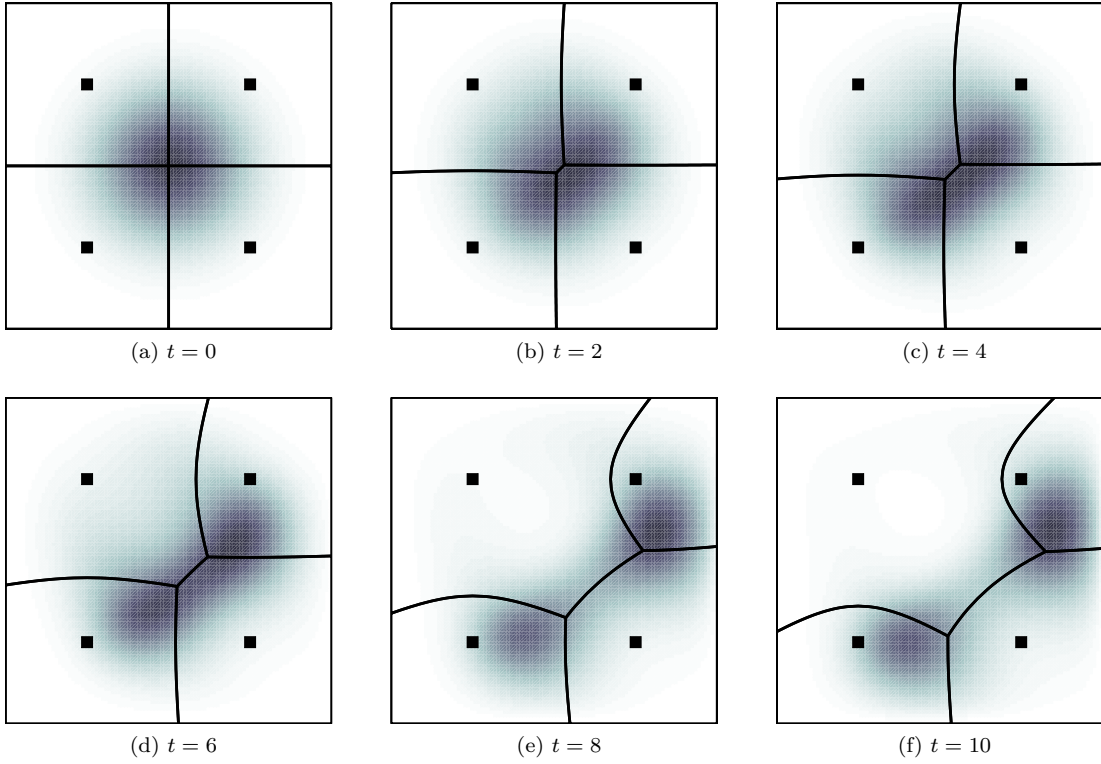


Figure 14: Advection over the unit square for $t \in [0, 10]$ and the optimal partitions $R_1^*(t), R_2^*(t), R_3^*(t), R_4^*(t)$.

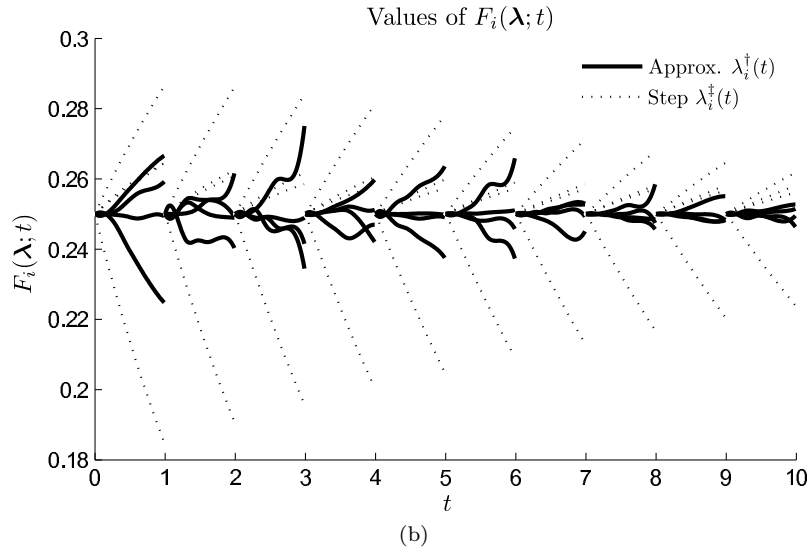
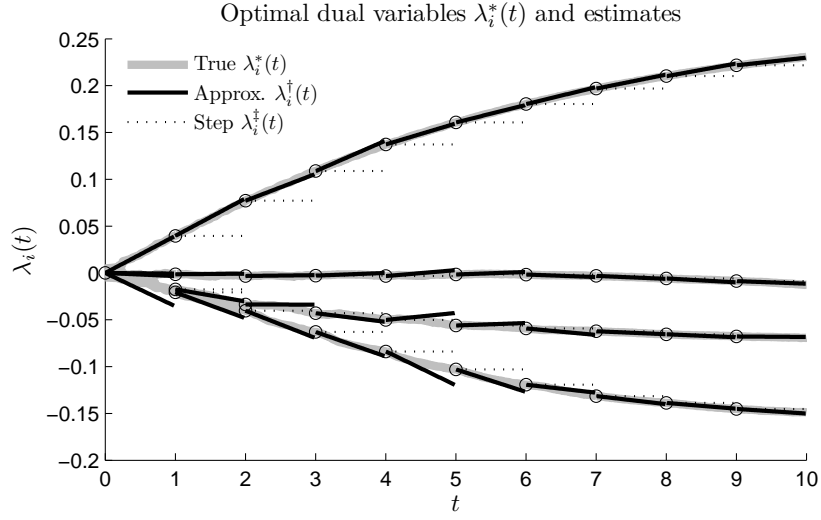


Figure 15: The optimal dual variables $\lambda_i^*(t)$ and their approximations $\lambda_i^{\dagger}(t)$ and $\lambda_i^{\ddagger}(t)$, and the resulting masses of the induced partitions.

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Online supplement to “Shadow prices in territory division”

A Proofs of Theorem 1 and 3

In order to prove Theorems 1 and 3, we find it helpful to first state an important result from Section 8.6 of [26]:

Theorem 13. (*Lagrange Duality*) Let f be a real-valued convex functional defined on a convex subset Ω of a vector space \mathfrak{X} , and let \mathfrak{G} be a convex mapping of \mathfrak{X} into a normed space \mathfrak{Z} . Suppose there exists $\mathfrak{x}_1 \in \mathfrak{X}$ such that $\mathfrak{G}(\mathfrak{x}_1) < \theta$, where θ denotes the zero element, and that $\mu_0 := \inf \{f(\mathfrak{x}) : \mathfrak{G}(\mathfrak{x}) \leq \theta, \mathfrak{x} \in \Omega\}$ is finite. Then

$$\inf_{\mathfrak{x} \in \Omega, \mathfrak{G}(\mathfrak{x}) \leq \theta} f(\mathfrak{x}) = \max_{\mathfrak{z}^* \geq \theta} \varphi(\mathfrak{z}^*)$$

where

$$\varphi(\mathfrak{z}^*) = \inf_{\mathfrak{x} \in \Omega} f(\mathfrak{x}) + \langle \mathfrak{G}(\mathfrak{x}), \mathfrak{z}^* \rangle,$$

and the maximum on the right is achieved by some $\mathfrak{z}_0^* \geq \theta$.

We will assume in this section, without loss of generality, that $u_i(x) > 0$ for all i and all $x \in R$.

A.1 Proof of Theorem 1

We find it helpful to begin our proof by first considering the *dual* problem (7), which we will prove is equivalent to the original problem (1). The reason that we prefer to do things in this order is because it is easier to verify that a bounded optimal solution to (7) actually exists, as demonstrated by the following:

Lemma 14. *A bounded optimal solution λ^* to problem (7) exists.*

Proof. Let $S = \iint_R f(x) \max_i u_i(x) dA$ denote the objective function value of (7) at $\lambda = \mathbf{0}$ and let $Q = -2 \max_i \iint_R f(x) |u_i(x)| dA$, which implies that $Q \leq S$. Note that for any indices j and k there exists a finite threshold m_{jk} such that, if $\lambda_j - \lambda_k \geq m_{jk}$, then $\iint_{R_j} f(x) dA \leq \epsilon$, where $\epsilon = 1/2n$. Let M' be the maximum of all such thresholds m_{jk} , let $M'' = 4(S - Q)$, and let $M = \max\{M', M''\}$.

Suppose that λ satisfies $\mathbf{q}^T \lambda = 0$ and $\|\lambda\|_\infty > (n-1)M$. Let $\lambda_j = \max_i \lambda_i > 0$ and $\lambda_k = \min_i \lambda_i < 0$; by definition, we must have $\lambda_j \leq (n-1)|\lambda_k|$ and $|\lambda_k| > M$. Let $R_j := \{x \in R : u_j(x) - \lambda_j \geq u_k(x) - \lambda_k\}$ and let $R_k = R \setminus R_j$. (Note that this is different from our usual definition of the regions R_i because we are disregarding all indices other than j and k .) The objective function value of (7) is then

$$\begin{aligned} \iint_R f(x) \max_i \{u_i(x) - \lambda_i\} dA &\geq \iint_R f(x) \max\{u_j(x) - \lambda_j, u_k(x) - \lambda_k\} dA \\ &= \iint_{R_j} f(x)(u_j(x) - \lambda_j) dA + \iint_{R_k} f(x)(u_k(x) - \lambda_k) dA \\ &= \iint_{R_j} f(x)u_j(x) dA - \lambda_j \underbrace{\iint_{R_j} f(x) dA}_{\leq 1/2n} + \iint_{R_k} f(x)u_k(x) dA - \lambda_k \underbrace{\iint_{R_k} f(x) dA}_{\geq 1-1/2n} \\ &\geq \underbrace{\iint_{R_j} f(x)u_j(x) dA}_{\geq -\iint_R f(x)|u_j(x)| dA} + \underbrace{\iint_{R_k} f(x)u_k(x) dA}_{\geq -\iint_R f(x)|u_k(x)| dA} - \frac{\lambda_j}{2n} + |\lambda_k| \left(1 - \frac{1}{2n}\right) \\ &\geq Q - \frac{\lambda_j}{2n} + |\lambda_k| \left(1 - \frac{1}{2n}\right) \geq Q - \frac{n-1}{2n} |\lambda_k| + |\lambda_k| \left(1 - \frac{1}{2n}\right) \\ &= Q + |\lambda_k|/2 \geq S \end{aligned}$$

and therefore λ has an objective value no better than that induced by the zero vector. Thus, we can assume without loss of generality that problem (7) is restricted to the compact set $\|\lambda\|_\infty \leq (n-1)M$, which completes the proof. \square

To prove Theorem 1, we find it helpful to use the alternate formulation (6) of (7), reproduced below:

$$\begin{aligned} \underset{\lambda, \sigma(\cdot)}{\text{minimize}} \quad & \sum_{i=1}^n q_i \lambda_i + \iint_R f(x) \sigma(x) dA \quad s.t. \\ & \sigma(x) \geq u_i(x) - \lambda_i \quad \forall i, x. \end{aligned} \quad (25)$$

We now apply Theorem 13: in problem (25), the optimization variables are λ and $\sigma(\cdot)$, so we let $\mathfrak{X} = \Omega = \mathbb{R}^n \oplus L_1$, where L_1 represents all functions $h(\cdot)$ defined on R such that $|h(x)|$ is Lebesgue integrable on R . We let $\mathfrak{f}(\mathfrak{r})$ be defined by

$$\mathfrak{f} : \begin{pmatrix} \lambda \\ \sigma(\cdot) \end{pmatrix} \mapsto \mathbf{q}^T \lambda + \iint_R f(x) \sigma(x) dA$$

and we let $\mathfrak{G} : \mathfrak{X} \rightarrow \mathfrak{Z}$ be defined by

$$\mathfrak{G} : \begin{pmatrix} \lambda \\ \sigma(\cdot) \end{pmatrix} \mapsto \begin{pmatrix} \xi_1(\cdot) - \lambda_1 \\ \vdots \\ \xi_n(\cdot) - \lambda_n \end{pmatrix}$$

where $\xi_i(x) := u_i(x) - \sigma(x)$, so that $\mathfrak{Z} = \underbrace{L_1 \oplus \cdots \oplus L_1}_n$. By the preceding existence argument for λ^* , we can replace the infimum operator of Theorem 13 with the minimum operator. From basic functional analysis, we have $\mathfrak{Z}^* = \underbrace{L_\infty \oplus \cdots \oplus L_\infty}_n$, where L_∞ denotes all *bounded* functions on R . Let $(J_1(\cdot), \dots, J_n(\cdot))$ denote an element of \mathfrak{Z}^* . Theorem 13 says that

$$\begin{aligned} & \min_{\mathfrak{r} \in \Omega, \mathfrak{G}(\mathfrak{r}) \leq \theta} \mathfrak{f}(\mathfrak{r}) \\ &= \max_{\mathfrak{z}^* \geq \theta} \left\{ \inf_{\mathfrak{r} \in \Omega} \mathfrak{f}(\mathfrak{r}) + \langle \mathfrak{G}(\mathfrak{r}), \mathfrak{z}^* \rangle \right\} \\ &= \max_{J_i(\cdot) \geq 0} \left\{ \inf_{\lambda, \sigma(\cdot)} \mathbf{q}^T \lambda + \iint_R f(x) \sigma(x) dA + \iint_R \sum_{i=1}^n J_i(x) (\xi_i(x) - \lambda_i) dA \right\} \\ &= \max_{J_i(\cdot) \geq 0} \left\{ \inf_{\lambda, \sigma(\cdot)} \left[\sum_{i=1}^n \left(q_i - \iint_R J_i(x) dA \right) \lambda_i \right] + \iint_R \left(\sum_{i=1}^n J_i(x) u_i(x) - J_i(x) \sigma(x) \right) + f(x) \sigma(x) dA \right\} \\ &= \max_{J_i(\cdot) \geq 0} \left\{ \inf_{\lambda, \sigma(\cdot)} \left[\sum_{i=1}^n \left(q_i - \iint_R J_i(x) dA \right) \lambda_i \right] + \iint_R \sigma(x) \left(f(x) - \sum_{i=1}^n J_i(x) \right) dA + \iint_R \sum_{i=1}^n J_i(x) u_i(x) dA \right\} \end{aligned}$$

Clearly we need $\iint_R J_i(x) dA = q_i$ for all i and $\sum_i J_i(x) = f(x)$ for all $x \in R$ (the infimum term over all λ and $\sigma(\cdot)$ is unbounded below otherwise). Introducing new functions $I_i(x) := J_i(x) / f(x)$, the above is equivalent to

$$\max_{I_i(\cdot) \geq 0} \left\{ \inf_{\lambda, \sigma(\cdot)} \left[\sum_{i=1}^n \left(q_i - \iint_R f(x) I_i(x) dA \right) \lambda_i \right] + \iint_R f(x) \sigma(x) \left(1 - \sum_{i=1}^n I_i(x) \right) dA + \iint_R \sum_{i=1}^n f(x) u_i(x) I_i(x) dA \right\}$$

so that $\iint_R f(x) I_i(x) dA = q_i$ for all i and $\sum_i I_i(x) = 1$ for all $x \in R$. Thus, the problem (7) and the problem

$$\begin{aligned} & \underset{I_1(\cdot), \dots, I_n(\cdot)}{\text{maximize}} \quad \iint_R \sum_{i=1}^n f(x) u_i(x) I_i(x) dA \quad s.t. \\ & \iint_R f(x) I_i(x) dA = q_i \quad \forall i \\ & \sum_{i=1}^n I_i(x) = 1 \quad \forall x \\ & I_i(x) \geq 0 \quad \forall i, x. \end{aligned}$$

are primal-dual pairs as desired, and by Theorem 13 we know that an optimal solution $I_1^*(\cdot), \dots, I_n^*(\cdot)$ to problem (1) exists. This completes the proof.

A.2 Proof of Theorem 3

To prove Theorem 3, we will again consider the dual problem (10) first and show that this is equivalent to problem (2). Using a very similar argument to that of Lemma 14, which we omit for brevity, it is not hard to verify that there must exist an optimal solution λ^* to problem (10). In order to apply Theorem 13 directly, we will alter problem (10) without loss of generality by substituting an inequality in the linear constraint on λ and by writing the problem as a linear program:

$$\begin{aligned} \text{minimize}_{\lambda, \sigma(\cdot)} \quad & \iint_R f(x) \sigma(x) dA \quad \text{s.t.} \\ & \sigma(x) \geq \lambda_i u_i(x) \quad \forall i, x \\ & \sum_{i=1}^n \lambda_i \geq 1 \\ & \lambda_i \geq 0 \quad \forall i \end{aligned} \tag{26}$$

In problem (26), the optimization variables are λ and $\sigma(\cdot)$, so we let $\mathfrak{X} = \mathbb{R}^n \oplus L_1$, where L_1 represents all functions $h(\cdot)$ defined on R such that $|h(x)|$ is Lebesgue integrable on R . Let Ω denote the positive orthant, i.e. $\lambda_i \geq 0$ and $\sigma(x) \geq 0$ for all $x \in R$. We let $\mathfrak{f}(\mathfrak{r})$ be defined by

$$\mathfrak{f} : \begin{pmatrix} \lambda \\ \sigma(\cdot) \end{pmatrix} \mapsto \iint_R f(x) \sigma(x) dA$$

and we let $\mathfrak{G} : \mathfrak{X} \rightarrow \mathfrak{Z}$ be defined by

$$\mathfrak{G} : \begin{pmatrix} \lambda \\ \sigma(\cdot) \end{pmatrix} \mapsto \begin{pmatrix} \xi_1(\cdot) \\ \vdots \\ \xi_n(\cdot) \\ 1 - \sum_i \lambda_i \end{pmatrix}$$

where $\xi_i(x) := \lambda_i u_i(x) - \sigma(x)$, so that $\mathfrak{Z} = \underbrace{L_1 \oplus \dots \oplus L_1}_n \oplus \mathbb{R}$. We can again replace the infimum operator in Theorem 13 with the minimum operator. Let $(J_1, \dots, J_n, t) \in \underbrace{L_\infty \oplus \dots \oplus L_\infty}_n \oplus \mathbb{R}$ denote an element of the dual space \mathfrak{Z}^* , and as before define $I_i(x) := J_i(x)/f(x)$. We then find that

$$\begin{aligned} & \min_{\mathfrak{r} \in \Omega, \mathfrak{G}(\mathfrak{r}) \leq \theta} \mathfrak{f}(\mathfrak{r}) \\ &= \max_{\mathfrak{z}^* \geq \theta} \left\{ \inf_{\mathfrak{r} \in \Omega} \mathfrak{f}(\mathfrak{r}) + \langle \mathfrak{G}(\mathfrak{r}), \mathfrak{z}^* \rangle \right\} \\ &= \max_{J_i(\cdot), t \geq 0} \left\{ \inf_{\lambda \geq 0, \sigma(\cdot) \geq 0} \iint_R f(x) \sigma(x) dA + \iint_R \sum_{i=1}^n J_i(x) \xi_i(x) dA + t \left(1 - \sum_{i=1}^n \lambda_i \right) \right\} \\ &= \max_{J_i(\cdot), t \geq 0} \left\{ \inf_{\lambda \geq 0, \sigma(\cdot) \geq 0} \iint_R f(x) \sigma(x) + \sum_{i=1}^n J_i(x) \xi_i(x) dA + \iint_R f(x) t \left(1 - \sum_{i=1}^n \lambda_i \right) dA \right\} \\ &= \max_{J_i(\cdot), t \geq 0} \left\{ \inf_{\lambda \geq 0, \sigma(\cdot) \geq 0} \iint_R \left(\sum_{i=1}^n J_i(x) \xi_i(x) - f(x) t \lambda_i \right) + f(x) \sigma(x) + f(x) t dA \right\} \\ &= \max_{J_i(\cdot), t \geq 0} \left\{ \inf_{\lambda \geq 0, \sigma(\cdot) \geq 0} \iint_R \sigma(x) \left(f(x) - \sum_{i=1}^n J_i(x) \right) + \left[\sum_{i=1}^n \lambda_i (J_i(x) u_i(x) - f(x) t) \right] + f(x) t dA \right\} \\ &= \max_{I_i(\cdot), t \geq 0} \left\{ \inf_{\lambda \geq 0, \sigma(\cdot) \geq 0} \iint_R \sigma(x) f(x) \left(1 - \sum_{i=1}^n I_i(x) \right) dA + \left[\sum_{i=1}^n \lambda_i \left(\iint_R f(x) u_i(x) I_i(x) dA - t \right) \right] + t \right\} \end{aligned}$$

which implies that $\sum_i I_i(x) \leq 1$ for all $x \in R$ and $\iint_R f(x) I_i(x) u_i(x) dA \geq t$ for all i . Thus, problem (26) and

the problem

$$\begin{aligned}
& \underset{t, I_1(\cdot), \dots, I_n(\cdot)}{\text{maximize}} \quad t && \text{s.t.} && (27) \\
& t \leq \iint_R f(x) u_i(x) I_i(x) dA \quad \forall i \\
& \sum_{i=1}^n I_i(x) \leq 1 \quad \forall x \\
& I_i(x) \geq 0 \quad \forall i, x
\end{aligned}$$

are primal-dual pairs. We see that problem (27) differs from the linear relaxation of problem (8) only by the inequality $\sum_{i=1}^n I_i(x) \leq 1$ for all x ; it is of course trivial to see that equality must hold at optimality for problem (27) which confirms that the two problems are equivalent. This completes the proof.

B Ambiguities arising due to duality

Section 4, and Algorithms 1 and 2, show how to solve problems (1) and (2) by way of complementary slackness: given the optimal Lagrange multiplier λ^* to either problem, we define each optimal region R_i^* to be those points $x \in R$ such that either $u_i(x) - \lambda_i^*$ or $\lambda_i^* u_i(x)$ is maximal over all i (depending on what problem we want to solve). For most of the examples used in this paper, this characterization is sufficient, because the set of “ambiguous points” x where this maximal index is not unique has measure zero. In this section, we show how to define the optimal partition when there exists a set with positive measure on which the maximal index i is not unique. As we noted in Section 5, one case where this arises is when R contains a set of obstacles and we set $u_i(x) = -d(x, p_i)$, where $d(x, p_i)$ is the length of the shortest path between x and p_i .

B.1 Ambiguities in (1)

Let λ^* denote an optimal Lagrange multiplier for problem (7), the dual of (4). Let R_1^+, \dots, R_n^+ denote the *strict dominance regions* where $u_i(x) - \lambda_i^*$ is strictly maximal for some i , and let R_1^-, \dots, R_k^- denote the *ambiguous dominance regions* where strict optimality does not hold. Associated with each ambiguous dominance region R_j^- is an index set $\mathcal{I}_j \subseteq \{1, \dots, n\}$ that indicates the set of indices i for which $u_i(x) - \lambda_i^*$ is maximal.

Recall that Theorem 13 guarantees that an optimal solution to the linear relaxation of (1), i.e. (4), must exist. Let $I_1^*(\cdot), \dots, I_n^*(\cdot)$ denote this optimal solution, which would of course be unknown to us. By Theorem 13, we are guaranteed that strong duality holds, i.e. that

$$\begin{aligned}
\text{OPT} &:= \iint_R f(x) \max_i \{u_i(x) - \lambda_i^*\} dA = \sum_{i=1}^n \iint_R f(x) u_i(x) I_i^*(x) dA \\
&= \left(\sum_{i=1}^n \iint_{R_i^+} f(x) u_i(x) dA \right) + \left(\sum_{j=1}^k \iint_{R_j^-} f(x) \sum_{i \in \mathcal{I}_j} u_i(x) I_i^*(x) dA \right).
\end{aligned}$$

Consider a particular ambiguous dominance region R_j^- ; by construction, we are guaranteed that $u_i(x) - \lambda_i^* = u_{i'}(x) - \lambda_{i'}^*$ for all $i, i' \in \mathcal{I}_j$ and all $x \in R_j^-$. Let $\bar{u}(x) = u_i(x) - \lambda_i^*$ for any (equivalently, all) $i \in \mathcal{I}_j$, so that

$$\begin{aligned}
\iint_{R_j^-} f(x) \sum_{i \in \mathcal{I}_j} u_i(x) I_i^*(x) dA &= \iint_{R_j^-} f(x) \sum_{i \in \mathcal{I}_j} (\bar{u}(x) + \lambda_i^*) I_i^*(x) dA \\
&= \iint_{R_j^-} f(x) \bar{u}(x) \underbrace{\sum_{i \in \mathcal{I}_j} I_i^*(x)}_{=1} dA + \iint_{R_j^-} f(x) \sum_{i \in \mathcal{I}_j} \lambda_i^* I_i^*(x) dA \\
&= \underbrace{\iint_{R_j^-} f(x) \bar{u}(x) dA}_{=: a_j \text{ (known)}} + \sum_{i \in \mathcal{I}_j} \lambda_i^* \iint_{R_j^-} f(x) I_i^*(x) dA.
\end{aligned}$$

Observe that the second term in the above, $\sum_i \lambda_i^* \iint_{R_j^-} f(x) I_i^*(x) dA$, does not actually depend on the *functions* $I_i^*(\cdot)$, but merely the amount of the mass in R_j^- that is allocated to i . In other words, setting $b_j = \iint_{R_j^-} f(x) dA$, we see that

$$\sum_{i \in \mathcal{I}_j} \lambda_i^* \iint_{R_j^-} f(x) I_i^*(x) dA = \sum_{i \in \mathcal{I}_j} \lambda_i^* \alpha_{ij} b_j$$

for some coefficients $\alpha_{ij} \geq 0$ such that $\sum_{i \in \mathcal{I}_j} \alpha_{ij} = 1$ for all j . Given an optimal Lagrange multiplier λ^* , and the dual optimal objective value OPT , it is therefore easy to find the optimal coefficients α_{ij} because we can write

$$\begin{aligned} \text{OPT} &= \iint_R f(x) \max_i \{u_i(x) - \lambda_i^*\} dA = \sum_{i=1}^n \iint_R f(x) u_i(x) I_i^*(x) dA \\ &= \underbrace{\left(\sum_{i=1}^n \iint_{R_i^+} f(x) u_i(x) dA \right)}_{=: c \text{ (known)}} + \left(\sum_{j=1}^k \iint_{R_j^-} f(x) \sum_{i \in \mathcal{I}_j} u_i(x) I_i^*(x) dA \right) \\ &= c + \sum_{j=1}^k \left(a_j + \sum_{i \in \mathcal{I}_j} \lambda_i^* \alpha_{ij} b_j \right) \end{aligned}$$

and solve for the terms α_{ij} using linear programming. It is then a trivial matter to divide each ambiguous region R_j^- into components with mass $\alpha_{ij} b_j$ in whatever manner we like.

B.2 Ambiguities in (2)

Let λ^* denote an optimal Lagrange multiplier for problem (10), the dual of the linear relaxation of (8). Define regions $R_1^+, \dots, R_n^+, R_1^-, \dots, R_k^-$, and $\mathcal{I}_1, \dots, \mathcal{I}_k$ analogously as in Section B.1. Theorem 13 again guarantees that an optimal solution to the linear relaxation of (8) must exist, so that

$$\begin{aligned} \text{OPT} &:= \iint_R f(x) \max_i \{\lambda_i^* u_i(x)\} dA = t^* = \iint_R f(x) u_i(x) I_i^*(x) dA \quad \forall i \\ &= \iint_{R_i^+} f(x) u_i(x) dA + \sum_{j: i \in \mathcal{I}_j} \iint_{R_j^-} f(x) u_i(x) I_i^*(x) dA \quad \forall i \end{aligned}$$

where we have used the fact that, at optimality, it must be true that $t^* = \iint_R f(x) u_i(x) I_i^*(x) dA$ for all i (this occurs because we have assumed that $f(x) > 0$ and $u_i(x) > 0$ for all i and $x \in R$, and thus $\lambda_i^* > 0$ for all i as well). Consider a particular ambiguous dominance region R_j^- ; by construction, we are guaranteed that $\lambda_i^* u_i(x) = \lambda_{i'}^* u_{i'}(x)$ for all $i, i' \in \mathcal{I}_j$ and all $x \in R_j^-$. Let $\bar{u}(x) = \lambda_i^* u_i(x)$ for any (equivalently, all) $i \in \mathcal{I}_j$, so that

$$\iint_{R_j^-} f(x) u_i(x) I_i^*(x) dA = \frac{1}{\lambda_i^*} \iint_{R_j^-} f(x) \bar{u}(x) I_i^*(x) dA.$$

We again observe that the term $\iint_{R_j^-} f(x) \bar{u}(x) I_i^*(x) dA$ does not actually depend on the *function* $I_i^*(\cdot)$ but merely the amount of the mass in R_j^- that is allocated to i . In other words, setting $b_j = \iint_{R_j^-} f(x) \bar{u}(x) dA$, we see that

$$\frac{1}{\lambda_i^*} \iint_{R_j^-} f(x) \bar{u}(x) I_i^*(x) dA = \frac{\alpha_{ij} b_j}{\lambda_i^*}$$

for some coefficients $\alpha_{ij} \geq 0$ such that $\sum_{i \in \mathcal{I}_j} \alpha_{ij} = 1$ for all j . Thus, given an optimal Lagrange multiplier λ^* , and the dual optimal objective value OPT , it is therefore easy to find the optimal coefficients α_{ij} because we can write

$$\begin{aligned} \text{OPT} &= \underbrace{\iint_{R_i^+} f(x) u_i(x) dA}_{c_i \text{ (known)}} + \sum_{j: i \in \mathcal{I}_j} \iint_{R_j^-} f(x) u_i(x) I_i^*(x) dA \quad \forall i \\ &= c_i + \frac{1}{\lambda_i^*} \sum_{j: i \in \mathcal{I}_j} \alpha_{ij} b_j \quad \forall i \end{aligned}$$

and solve for the terms α_{ij} using linear programming. It is then a trivial matter to divide each ambiguous region R_j^- into components with mass $\alpha_{ij}b_j$ in whatever manner we like.