Example: 9-qubit Shor code

composition of
$$\begin{array}{ccc} Z & Z & 1 \\ 1 & Z & Z \\ \hline Z & Z & Z \\ X & X & X \end{array} \quad \text{with} \quad \begin{array}{ccc} X & X & 1 \\ 1 & X & X \\ \hline Z & Z & Z \\ X & X & X \end{array} \quad \rightarrow \quad \begin{array}{ccccccccc} Z & Z & 1 & & & & & & \\ 1 & Z & Z & & & & & & \\ & & & Z & Z & 1 & & & \\ & & & 1 & Z & Z & & & \\ & & & & & & Z & Z & 1 \\ & & & & & & 1 & Z & Z \\ X & X & X & X & X & X & 1 & 1 & 1 \\ 1 & 1 & 1 & X & X & X & X & X & X \end{array}$$
$$\bar{Z} = ZZZZZZZZZ$$
$$\bar{X} = XXXXXXXXX$$

[[9,1,3]] CSS, not self dual

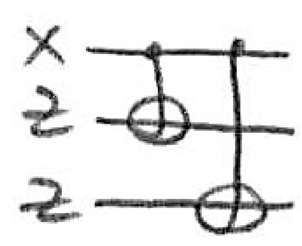
correcťs ≤ 1 X error in every block of 3

and ≤ 1 Z error total

Ex: $|\bar{0}\rangle = ?$
$|\bar{+}\rangle = ?$

Note: $|\bar{+}\rangle$ stab'd by $\begin{array}{ccc} Z & Z & 1 \\ 1 & Z & Z \\ X & X & X \end{array}$ on each block

$\Rightarrow |\bar{+}\rangle = (|000\rangle + |111\rangle)^{\otimes 3}$

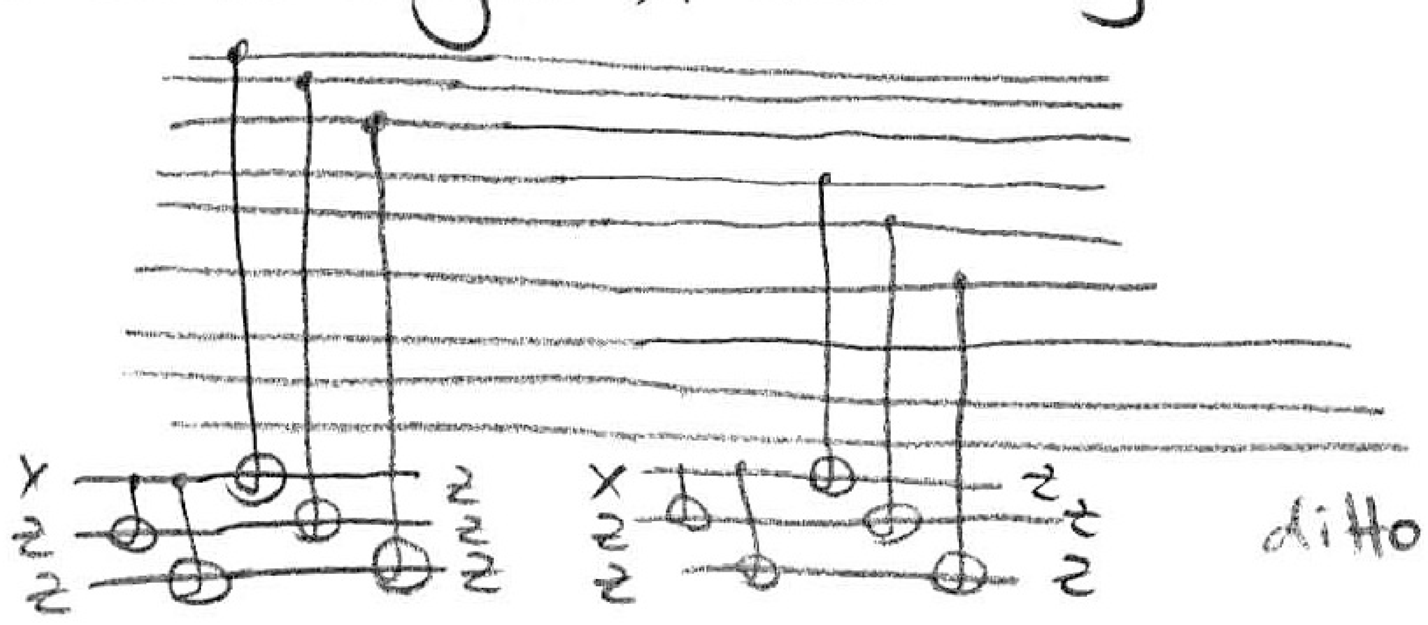


prepares a block

— automatically fault tolerant

(every error equivalent to a weight-one fault)

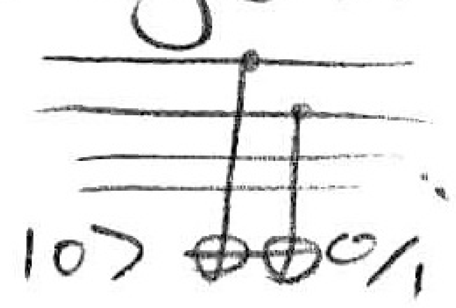
$\Rightarrow$ Steane-style X EC easy



ditto

- don't need to verify $|\bar{+}\rangle$
- don't need to keep it in memory

Steane-style X EC even easier



meas. ZZ

- one-qubit ancilla only

~ but Z error correction still difficult

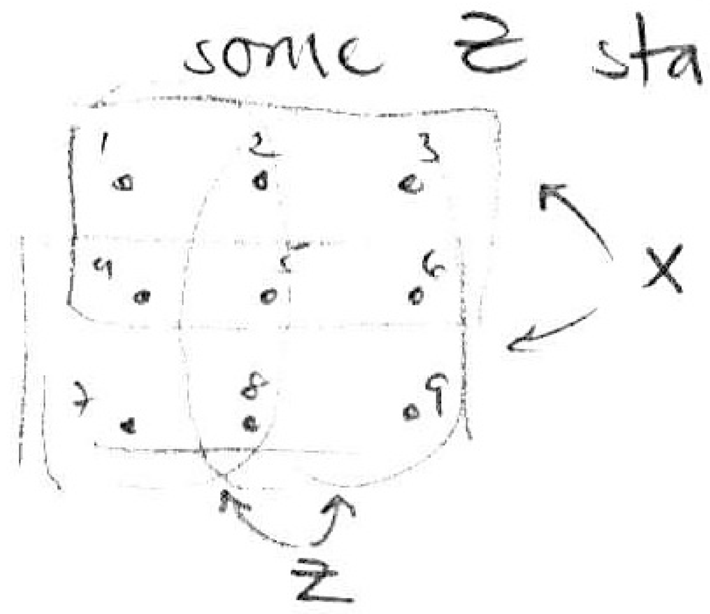
?: Can we simplify Z EC & make X, Z errors symmetric?

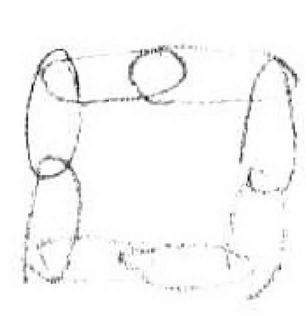
- note $H^{\otimes 9}$ doesn't work to swap X, Z

<u>Subsystem stabilizer codes</u>

- since correcting more X errors than needed, can throw away some Z stabilizers, get down to

$$\begin{array}{ccccccccc} X & X & X & X & X & X & 1 & 1 & 1 \\ 1 & 1 & 1 & X & X & X & X & X & X \\ Z & Z & 1 & Z & Z & 1 & Z & Z & 1 \\ 1 & Z & Z & 1 & Z & Z & 1 & Z & Z \\ \hline \end{array}$$





$$\begin{array}{rl} \bar{X}_1 = & X X X \; X X X \; X X X \\ \bar{Z}_1 = & Z Z Z \; Z Z Z \; Z Z Z \\ \bar{X}_2 = & X 1 1 \; X 1 1 \; 1 1 1 \\ \bar{Z}_2 = & Z Z 1 \; 1 1 1 \; 1 1 1 \\ \bar{X}_3 = & 1 1 X \; 1 1 X \; 1 1 1 \\ \bar{Z}_3 = & 1 Z Z \; 1 1 1 \; 1 1 1 \\ \bar{X}_4 = & 1 1 1 \; X 1 1 \; X 1 1 \\ \bar{Z}_4 = & 1 1 1 \; 1 1 1 \; Z Z 1 \\ \bar{X}_5 = & 1 1 1 \; 1 1 X \; 1 1 X \\ \bar{Z}_5 = & 1 1 1 \; 1 1 1 \; 1 Z Z \end{array}$$

"Bacon-Shor code"
also Knill & Poulin

Remark 1: $X \leftrightarrow Z$ symmetrical

$\bar{H} = H^{\otimes 9}$ followed by a transposition of the qubits

Remark 2:

$d = 2$, but first qubit is protected to distance 3!

i.e. any nontrivial operator acting nontrivially on qubit 1 has weight ≥ 3.

Proof: Eg. can detect the block a Z error falls into and correct it, only causing a logical error on some other logical qubit ✓

QECC: $C \subseteq \mathcal{H}_{2^n}$

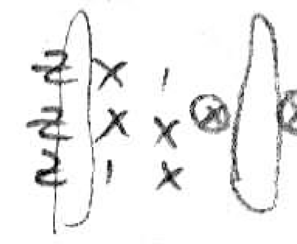
subsystem QECC: $(C \otimes C') \subseteq \mathcal{H}_{2^n}$

don't care about these qubits

$|\bar{\psi}\rangle \otimes |\bar{\psi}'\rangle$ typical codeword

$\Rightarrow$ for X EC, use $|\bar{+}\,\bar{0}\bar{0}\bar{0}\bar{0}\rangle = (|000\rangle + |111\rangle)^{\otimes 3}$

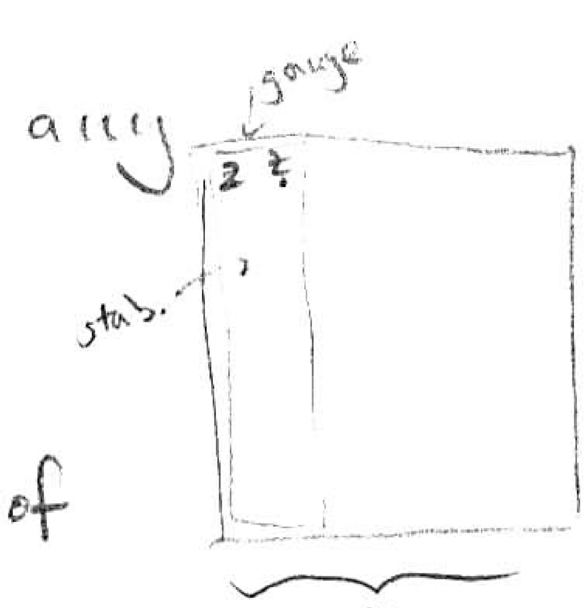
→ single-qubit syndrome extraction

$\Rightarrow$ for Z EC, use $|\bar{0}\,\bar{+}\bar{+}\bar{+}\bar{+}\rangle =$  $= \left(|{+}{+}{+}\rangle + |{-}{-}{-}\rangle\right)^{\otimes 3}$

→ single-qubit syndrome extraction

Remark 3: Bacon's construction trivially generalizes to work for any two classical linear codes. (eg. 25-qubit B-S code, ...)

4-qubit B-S simplest..



$n = m^2$, $2(m-1)$ stabs.

$(m-1)^2$ gauge qubits

$\Rightarrow m^2 - (m-1)^2 - 2(m-1) = 1$ encoded qubit

$d = m$

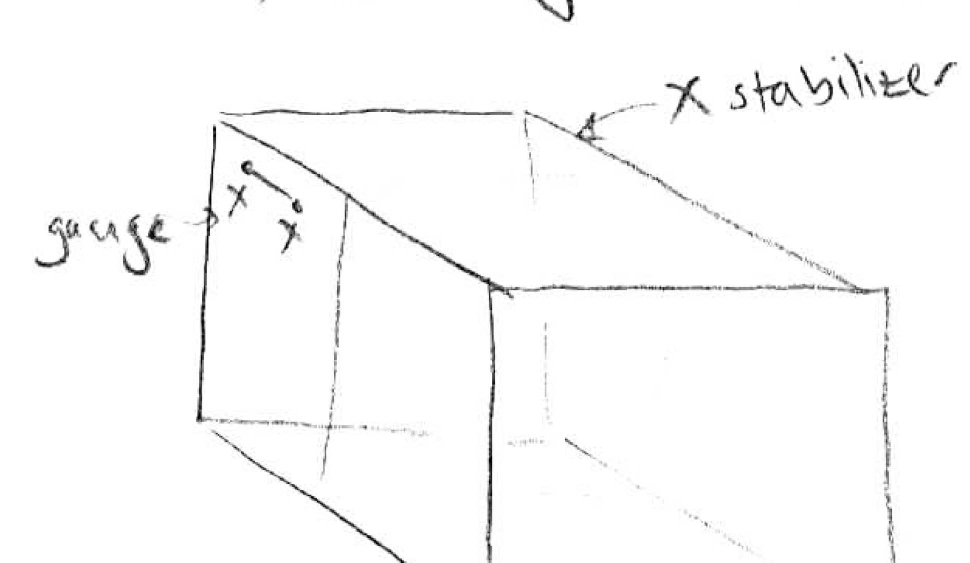
Remark 4: According to numerical diagonalization, ground space of

$$H = -X_1X_4 - X_2X_5 - X_3X_6 - X_4X_7X_5X_8 - X_6X_9$$
$$-Z_1Z_2 - Z_2Z_3 - Z_4Z_5 - Z_5Z_6 - Z_7Z_8 - Z_8Z_9$$

is spanned by $|\bar{0}_1 \bar{+}_{2345}\rangle$, $|\bar{1}_1 \bar{+}_{2345}\rangle$.

$\Rightarrow$ codespace is ground space of a local Hamiltonian

In 3D, the "compass model" with YY terms in the third direction, is conjectured to be self-stabilizing



$n = m^3$, $3(m-1)$ stabilizers

$(m^2-1)(m-1)$ gauge qubits

$\Rightarrow 2 + m(m-2)$ logical ?