QIC 710 Lecture 13 Kraus representation theorem, POVMs, general state distinguishing

<u>Def</u>: A general quantum operation $\mathcal{E}$ is a
- linear map, that
- preserves Hermiticity,
- preserves trace, and
- is completely positive, ie.

  $\mathcal{E} \otimes \mathbb{1}$ preserves positivity for all extensions.

<u>Example</u>: The transpose map is positive (leaves eigenvalues unchanged) but <u>not</u> completely positive.

on $\sum_x |x\rangle \otimes |x\rangle$ maximally entangled state

$$\underbrace{(T}_{\text{transpose}} \otimes \mathbb{1})\left[\sum_{xy} |x\rangle\langle y| \otimes |x\rangle\langle y|\right]$$

$$= \sum_{xy} |y\rangle\langle x| \otimes |x\rangle\langle y|$$

e.g. in 2D, $\begin{pmatrix} 1&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&1 \end{pmatrix} \longrightarrow |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|$

a swap operation $\begin{pmatrix} 1&&& \\ &0&1& \\ &1&0& \\ &&&1 \end{pmatrix}$

<u>not</u> positive semi-definite

<u>Theorem</u> (Kraus representation theorem): Any superoperator satisfying the above four conditions has a Kraus representation (and therefore can be implemented via Stinespring dilation). with at most $(\dim \mathcal{H})^2$ Kraus operators.

Thus we have three fully equivalent definitions of general quantum operations:
1. completely positive superoperator
2. Kraus representation
3. $\mathrm{Tr}_B\, U(\rho \otimes |0\rangle\langle 0|)U^+$

More examples: depolarizing noise, partial trace, spin relaxation, composition of two superoperators, general quantum measurements ...

note: superoperators are generally <u>not</u> invertible (unless unitary)

Kraus Representation Theorem

for a proof see [Preskill lecture notes, Ch. 3.2]

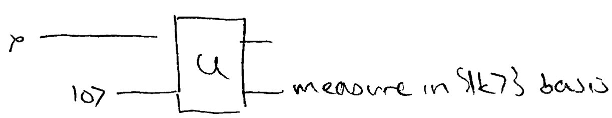
Main idea: the "Jamiolkowski isomorphism" (or Choi-J.)

Let $|\Psi\rangle = \sum_{i=1}^{N} |i\rangle \otimes |i\rangle$.

Then a superoperator $\mathcal{E}$ is characterized by

$$(\mathcal{E}\otimes \mathbb{1})(|\Psi\rangle\langle\Psi|)$$
$$= (\mathcal{E}\otimes \mathbb{1})\left(\sum_{ij} |i\rangle\langle j| \otimes |i\rangle\langle j|\right)$$
$$= \sum_{ij} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j|$$
≥ 0 since $\mathcal{E}$ is completely positive

General (open-system) measurements (POVMs)



$$\mathbb{P}[k] = \mathrm{Tr}\left[(\mathbb{1}\otimes |k\rangle\langle k|)\, U(\rho\otimes |0\rangle\langle 0|)U^\dagger\right]$$
$$= \sum_i \langle i|\langle k|\, U(\rho\otimes|0\rangle\langle 0|)U^\dagger\, |i\rangle|k\rangle$$
$$= \sum_i \langle i|\, M_k \rho M_k^\dagger\, |i\rangle \qquad \text{where } M_k = (\mathbb{1}\otimes\langle k|)\,U(\mathbb{1}\otimes|0\rangle)$$
$$= \mathrm{Tr}(\rho M_k^\dagger M_k)$$

and conditioned on outcome k, the state is

$$\rho' = \frac{1}{\sqrt{\mathbb{P}[k]}}\cdot(\mathbb{1}\otimes\langle k|)\,U(\rho\otimes|0\rangle\langle 0|)U^\dagger(\mathbb{1}\otimes|k\rangle)$$
$$= \frac{1}{\sqrt{\mathbb{P}[k]}}\, M_k \rho M_k^\dagger$$

Def: A measurement is specified by operators M_k with $\sum_k M_k^\dagger M_k = \mathbb{1}$.

On a state ρ, the probability of measuring k is

$$\mathbb{P}[k] = \mathrm{Tr}(\rho M_k^\dagger M_k)$$

and the new state is then given by

$$\rho' = M_k \rho M_k^\dagger / \sqrt{\mathbb{P}[k]}.$$

Def POVM (positive operator-valued measurement):

$$E_k \geq 0, \quad \sum_k E_k = \mathbb{1} \qquad \mathbb{P}[k] = \mathrm{Tr}(\rho E_k)$$

$E_k = M_k^\dagger M_k$ works and conversely $M_k = \sqrt{E_k}$ implements the POVM.

Examples... Notice: POVM elements need not be orthogonal.

Keep in mind: Church of the Larger Hilbert Space

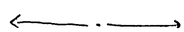
Classic example. Optimal distinguishing measurements for symmetrical set of single-qubit pure states

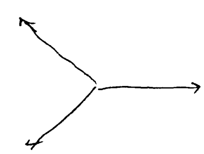
Assume we are given one of the states $|\psi_1\rangle, \ldots, |\psi_k\rangle$, with equal probabilities $1/k$, and our goal is to guess which one.

In general, there is no closed-form expression for the optimal distinguishing POVM. But if the states satisfy a symmetry, then the optimal POVM should satisfy the same symmetry (proof?), so we can find it.

Examples:

$\{|0\rangle, |1\rangle\}$ $\qquad$ $\{|0\rangle, \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\}$



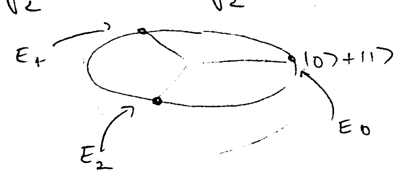


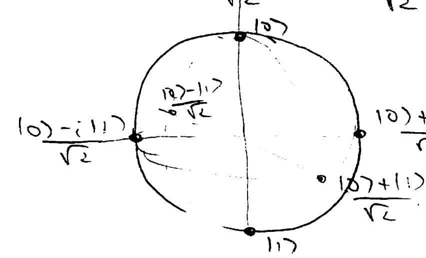
$\{|0\rangle + e^{\frac{2\pi i j}{k}}|1\rangle : j = 0, 1, \ldots, k-1\}$ $\qquad$ Pauli eigenstates

$|0\rangle, |1\rangle, \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}, \frac{|0\rangle \pm i|1\rangle}{\sqrt{2}}$

$k=3$

$\frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle + e^{\pm\frac{2\pi i}{3}}|1\rangle}{\sqrt{2}}$





A measurement using projections can only have two outcomes, and will not be optimal. Instead, use a 3-outcome POVM E_0, E_1, E_2.

We want to maximize over 2×2 positive semi-definite matrices satisfying $E_0 + E_1 + E_2 = \mathbb{1}$, $\quad \frac{1}{3}\sum_{j=0}^{2} \mathrm{Tr}\left[E_j \cdot \frac{1}{2}\begin{pmatrix} 1 & e^{-2\pi i j/3} \\ e^{2\pi i j/3} & 1 \end{pmatrix}\right]$

Expressed in the Bloch sphere coordinates

$$|+\rangle\langle +| = \tfrac{1}{2}(\mathbb{1} + X), \quad \tfrac{1}{2}(|0\rangle + e^{2\pi i/3}|1\rangle)(\langle 0| + e^{-\pi i/3}\langle 1|) = \tfrac{1}{2}(\mathbb{1} - \tfrac{1}{2}X + \tfrac{\sqrt{3}}{2}Y)$$

The POVM elements can also be expressed in their Pauli coordinates, and by symmetry the Z coordinates should be 0, while the X and Y coordinates should be in the same proportion as above. We want E_0, E_1, E_2 to be as close as possible to the above projections. Set them to be $\frac{2}{3}$ the above projections.

Definition: The Belavkin-Hausladen-Wooters "pretty good measurement" (PGM) is given by

$$E_k^{PGM} = \left(\sum_j \rho_j\right)^{-1/2} \rho_k \left(\sum_j \rho_j\right)^{-1/2}$$

Theorem [Barnum & Knill, quant-ph/0004088]

$$P_{success}(E^{PGM}) \geq P_{success}(\text{optimal POVM})^2 .$$

Corollary: The failure rates satisfy

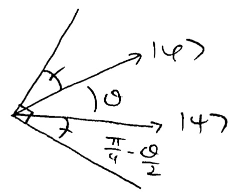
$$\begin{aligned} P_{fail}(E^{opt}) \leq P_{fail}(E^{PGM}) &\leq (1 - P_{success}(E^{opt})^2) \\ &= (1 + P_{success}(E^{opt}))\, P_{fail}(E^{opt}) \\ &\leq 2 P_{fail}(E^{opt}) . \end{aligned}$$

$\Rightarrow$ For states that are reasonably distinguishable, the pretty good measurement is approximately optimal. (see also [Tyson, 0907.3386])

Distinguishing quantum states:

Recall: For two pure states $|\varphi\rangle$ and $|\psi\rangle$, the optimal distinguishing measurement, i.e., the measurement that maximizes

$$\min\{\mathbb{P}[\text{say } |\varphi\rangle \mid |\varphi\rangle], \mathbb{P}[\text{say } |\psi\rangle \mid |\psi\rangle]\}$$



achieves

$$\begin{aligned}
&= \cos^2\left(\tfrac{\pi}{4}-\tfrac{\theta}{2}\right) \\
&= \tfrac{1}{2}\left(1+\cos\left(\tfrac{\pi}{2}-\theta\right)\right) \\
&= \tfrac{1}{2}(1+\sin\theta) \\
&= \tfrac{1}{2}\left(1+\sin\left(\cos^{-1}|\langle\varphi|\psi\rangle|\right)\right) \\
&= \tfrac{1}{2}\left(1+\sqrt{1-|\langle\varphi|\psi\rangle|^2}\right)
\end{aligned}$$

What about the general problem, of distinguishing two states ρ and σ?
- important for a notion of distance:
  states that are close together should be difficult to distinguish

Easy cases:
1. ρ and σ are pure — see above
2. ρ and σ can be simultaneously diagonalized,

i.e. $[\rho, \sigma]=0$, i.e. in some basis $\rho = \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{pmatrix}$ $\sigma = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}$

simply probability distributions

Optimal measurement samples and outputs the more likely distribution

$$\begin{aligned}
\mathbb{P}[\text{correctly identifies } \rho] &= \mathbb{P}[\text{correctly identifies } \sigma] \\
&= \tfrac{1}{2} + \tfrac{1}{4}\sum_j |p_j - q_j| \\
&= \tfrac{1}{2}\left(1 + TV(\{p_j\},\{q_j\})\right)
\end{aligned}$$

total variation distance

when each has 50% prior

Distance measures between quantum states

Euclidean distance $\| |\psi\rangle - |\varphi\rangle \|$

Fidelity $|\langle\varphi|\psi\rangle|$ usually preferable for pure states because it does not depend on the global phase

For mixed states ρ, σ, can also use Euclidean distance and fidelity (defined as $\mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$)

or the trace norm

$$\|M\|_{tr} = \mathrm{Tr}\,|M| = \mathrm{Tr}\sqrt{M^{\dagger}M}$$

the 1-norm of the eigenvalues

trace distance $\|\rho - \sigma\|_{tr}$

<u>Theorem</u>: For any two quantum states ρ and σ, the optimal measurement procedure for distinguishing between them succeeds with probability

$$\frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_{tr}.$$