11/18/10

QIC 710 Lecture 20 Quantum error correction

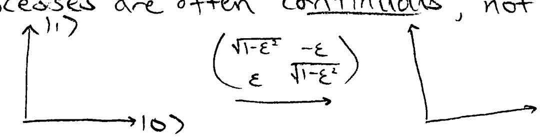
Problems for implementing a quantum computer:

1. Quantum entangled states are fragile

   e.g., measuring a single qubit of $|0^n\rangle + |1^n\rangle$, the cat state, collapses it entirely

   Similarly $\boxed{\text{cat}}$ interacting with a single photon and then tracing it out

2. Error processes are often <u>continuous</u>, not discrete



$$\begin{pmatrix} \sqrt{1-\varepsilon^2} & -\varepsilon \\ \varepsilon & \sqrt{1-\varepsilon^2} \end{pmatrix}$$

   and small errors can accumulate

3. There is no generic method for duplicating quantum information

   No cloning theorem: There is no linear map (let alone an isometry) that maps $|\psi\rangle \longmapsto |\psi\rangle \otimes |\psi\rangle$ for all $|\psi\rangle$

   Proof: If $|0\rangle \longmapsto |0\rangle\otimes|0\rangle$ and $|1\rangle \longmapsto |1\rangle\otimes|1\rangle$, then linearity requires $\alpha|0\rangle + \beta|1\rangle \longmapsto \alpha|00\rangle + \beta|11\rangle$

   $$\neq (\alpha|0\rangle + \beta|1\rangle)\otimes(\alpha|0\rangle + \beta|1\rangle) \quad \square$$

   (can't make backups!)

4. We want to <u>compute</u> on the data, not just protect it in memory

   Computations themselves need to be protected / "fault tolerant"

5. Noise rates in controllable quantum systems are usually quite high

   e.g. [1009.2267] compares some systems (environmentally accessible)

   superconducting charge qubit 1% noise for one-qubit ops

   10% noise for two-qubit ops

   classical devices have lower noise:

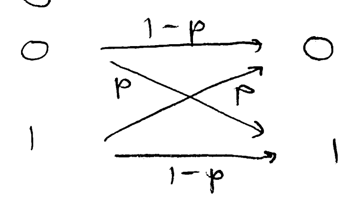
   transistor noise rate 10^{-16}?

   neuron noise rate ?

   ⋮

Codes for data memory or transmission

Binary symmetric channel / bit-flip noise



protect against using 3-bit repetition code (simplest example)

encode	transmit	decode
$0 \to 000$	$\longrightarrow$	$abc \to \text{majority}(a,b,c)$
$1 \to 111$		

$$\mathbb{P}[\text{erroneous transmission}] = 3p^2(1-p) + p^3$$
$$= 3p^2 - 2p^3$$
$$< p \text{ for } p < 1/2 \quad ✓$$

Quantum bit-flip channel (p)

With prob. $1-p$ apply $\mathbb{1}$, and with prob. p apply $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\mathcal{E}(\rho) = (1-p)\rho + p X \rho X .$$

The <u>same</u> code can be used, with care.

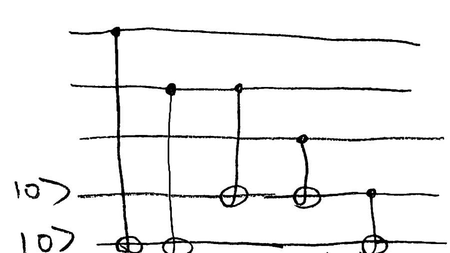
$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{encoding}} \alpha|000\rangle + \beta|111\rangle$$

e.g. if there is an error on the third qubit

$$\xrightarrow{\text{transmission}} \alpha|001\rangle + \beta|110\rangle$$

Decoding: – <u>cannot</u> measure and apply majority

$$\longrightarrow \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \text{ collapsed superposition}$$



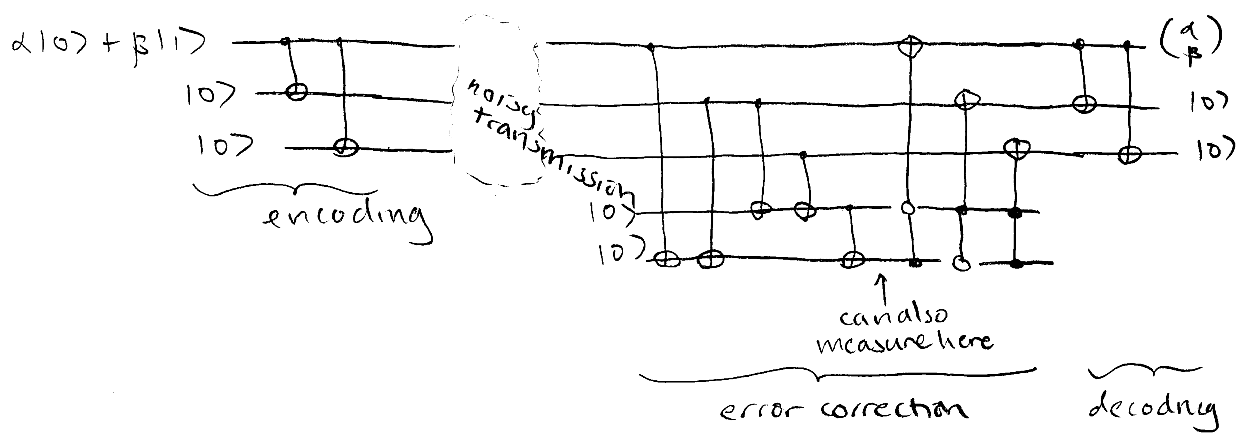
parity of 1st two qubits

$$(\alpha|001\rangle + \beta|110\rangle)|00\rangle \mapsto \alpha|001\rangle|11\rangle + \beta|110\rangle|11\rangle$$
$$= (\alpha|001\rangle + \beta|110\rangle)\underbrace{|11\rangle}_{\text{"syndrome"}}$$

Classical state	Syndrome
000	00
001	11
010	10
011	01
100	01
101	10
110	11
111	00

- no errors $\Rightarrow$ syndrome is 00
- X on qubit j $\Rightarrow$ syndrome is j in binary

$\Rightarrow$ apply a correction conditioned on the syndrome

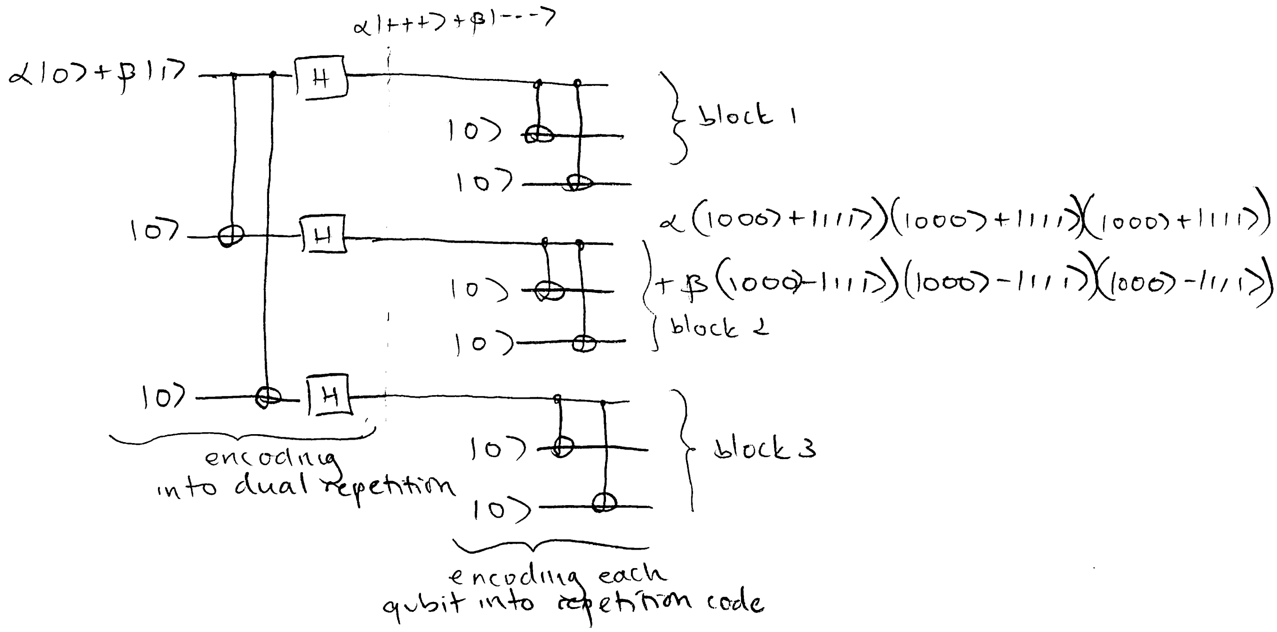


Correcting both phase and bit-flip errors

- use both the previous codes, i.e., "concatenate" them

since $Z_3(\alpha|000\rangle + \beta|111\rangle) = \alpha|000\rangle - \beta|111\rangle$

a phase-flip on the encoded states $|000\rangle, |111\rangle$, we can protect against it with another layer of coding



Shor's 9-qubit code:

Can correct up to one X error in each block, and up to one Z error total.

$\Rightarrow$ can protect against any of the errors X, Z or $Y = iXZ$ on a single qubit

(Note: Can correct errors without decoding)

Example: Apply XZ to qubit 4 of the state ...

Phase-flip channel ("dephasing", "T_2")

$$\mathcal{E}(\rho) = (1-p)\rho + p Z \rho Z \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|b\rangle = (-1)^b |b\rangle$$

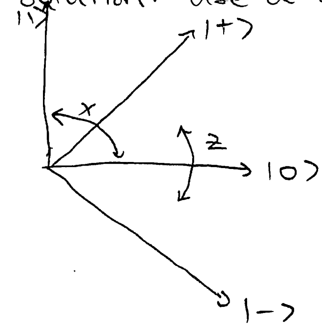
Observe: The above repetition code makes phase errors worse!

a Z on any one qubit will map

$$\alpha|000\rangle + \beta|111\rangle \longmapsto \alpha|000\rangle - \beta|111\rangle$$

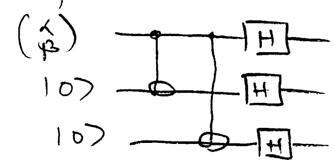
i.e., apply a logical phase flip.

Simple solution: Use a repetition code in the dual basis



encode $\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle$

using



phase flips now act the same as bit flips did before

$\Rightarrow$ Bob can apply Hadamards and then the same correction and decoding circuits

General quantum errors

Key point: It suffices for a quantum code to correct against the Pauli errors, because I, X, Y, Z form a <u>basis</u> for all 2×2 matrices.

Example: Let U be a 2×2 unitary

$$U = aI + bX + cY + dZ$$

Let $|\Psi\rangle$ be an encoded quantum state

$$U^{(j)}|\Psi\rangle = a|\Psi\rangle + bX^{(j)}|\Psi\rangle + cY^{(j)}|\Psi\rangle + dZ^{(j)}|\Psi\rangle$$

$$\xrightarrow[\text{extraction}]{\text{syndrome}} \begin{aligned} & a|\Psi\rangle \otimes |I \text{ syndrome}\rangle \\ & + bX^{(j)}|\Psi\rangle \otimes |X^{(j)} \text{ syndrome}\rangle \\ & + cY^{(j)}|\Psi\rangle \otimes |Y^{(j)} \text{ syndrome}\rangle \\ & + dZ^{(j)}|\Psi\rangle \otimes |Z^{(j)} \text{ syndrome}\rangle \end{aligned}$$

$$\xrightarrow[\text{correction}]{} |\Psi\rangle \otimes \begin{pmatrix} a|I \text{ syndrome}\rangle + b|X^{(j)} \text{syndrome}\rangle \\ + c|Y^{(j)} \text{syndrome}\rangle + d|Z^{(j)} \text{syndrome}\rangle \end{pmatrix}$$

Having recovered the codeword $|\Psi\rangle$, we can discard the extra ancillas.

More generally, consider an arbitrary quantum map (CPTP map)

$$\mathcal{E}^{(j)}(\rho) = \sum_{k=1}^{m} A_k^{(j)} \rho (A_k^{(j)})^\dagger$$

Each Kraus operator can be expanded in the Pauli basis

$$A_k = a_k I + b_k X + c_k Y + d_k Z$$

For notational simplicity let $\sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z$,

$$A_k = \sum_{p=0}^{3} a_{k,p}\sigma_p$$

$$\Rightarrow \quad \mathcal{E}^{(j)}(|\Psi\rangle\langle\Psi|) = \sum_{k=1}^{m}\sum_{p=0}^{3}\sum_{p'=0}^{3} a_{k,p}\,\sigma_p^{(j)}|\Psi\rangle\langle\Psi|\sigma_{p'}^{(j)}\,\overline{a_{k,p'}}$$

$$\xrightarrow[\text{extraction}]{\text{syndrome}} \sum_{k=1}^{m}\sum_{p=0}^{3}\sum_{p'=0}^{3} a_{k,p}\overline{a_{k,p'}}\left(\sigma_p^{(j)}|\Psi\rangle\langle\Psi|\sigma_{p'}^{(j)}\right)\otimes|\sigma_p^{(j)} \text{syn.}\rangle\langle\sigma_{p'}^{(j)} \text{syn.}|$$

$$\xrightarrow{\text{correction}} |\Psi\rangle\langle\Psi| \otimes (\text{stuff})$$

Observe: <u>Measuring</u> the syndrome would give

$$\sum_k \sum_p |a_{k,p}|^2 \, \sigma_p^{(j)}|\Psi\rangle\langle\Psi|\sigma_p^{(j)} \otimes |\sigma_p^{(j)} \text{syn.}\rangle\langle\sigma_p^{(j)} \text{syn.}|$$

→ thus it projects or collapses the arbitrary (perhaps continuous) error into just one of four <u>discrete</u> errors.

Codes that correct discrete I, X, Y, Z errors also correct continuous errors, eg $\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$, automatically, for free!