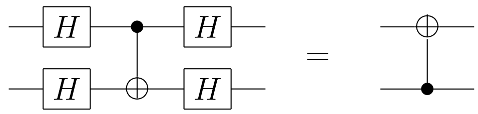
QIC 710 Problem Set 2

1. Write the 4×4 matrix of the unitary operation on two qubits resulting from performing a Hadamard transform H on the first qubit and a phase flip Z on the second qubit. Show your work.

2. Consider a CNOT gate whose second input is $|0\rangle - |1\rangle$. Describe the action of the CNOT gate on the first qubit.

Now show that if the CNOT gate is applied in the Hadamard basis, i.e., apply the Hadamard gate to the inputs and outputs of the CNOT gate, then the result is a CNOT gate with the control and target qubit swapped. That is, show



3. Show how to implement the conditional rotation given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix},$$

with $0 \leq \theta < 2\pi$, using CNOTs and single-qubit gates.

4. Consider a bipartite quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ shared between Alice and Bob. Prove that if Alice performs an arbitrary unitary operation on her part of $|\psi\rangle$ and then Bob measures his qubits in the standard basis, then the distribution of Bob's measurement is independent of Alice's actions.

5. Random walks and quantum walks:

(a) Consider a random walk on the n-vertex cycle. At each step, you flip a coin to decide whether to go left or right; the stochastic transition matrix is $A = \frac{1}{2}\sum_{j=0}^{n-1}(|j-1 \mod n\rangle + |j+1 \mod n\rangle)\langle j|$. Using your favorite numerical program, plot the distribution after 50 steps with $n = 100$, starting from 0.

(b) Consider a "*quantum walk*" on the same cycle. A quantum walk should be *reversible*, so instead of randomizing the coin at every time step, apply a Hadamard gate $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ to the coin. Then step left or right, depending on the coin. The state of the system thus is a vector in $\mathbf{C}^n \otimes \mathbf{C}^2$, where the first register stores the position and the second register the coin. Plot the distribution of position after 50 steps with $n = 100$, starting from $|0\rangle \otimes |0\rangle$.

(c) What main differences do you see between the plots in (a) and (b), qualitatively? In particular, approximately how far from the origin is the walk expected to have traveled? Extrapolate (guess) what happens for larger n. Is this interesting? Why or why not?

Note: Good programs for this include Mathematica, Matlab, Octave, Sage, numpy and Scilab. All but the first two are free. In Matlab or Octave, kron(A, B) implements the tensor product (A.K.A. Kronecker product) between matrices A and B. In Mathematica, KroneckerProduct[A,B], in numpy, numpy.kron(A, B), in SciLab kron(A,B) or A.*.B works.

Extra exercises—answers will not be graded:

6. Generalize quantum teleportation to qutrits: $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle$. Hint: Use a Fourier-transform gate F_3 in place of the Hadamard gate. F_3 maps the basis states of a qutrit as follows: $|j\rangle \mapsto \sum_k \omega^{jk}|k\rangle$, where $\omega = e^{2\pi i/3}$ is the principal third root of unity. Also use the sum modulo 3 gate in place of the controlled NOT: this gate maps $|j\rangle \otimes |k\rangle$ to $|j\rangle \otimes |j + k \mod 3\rangle$.

7. Prove the **No-Cloning Theorem**: There is no procedure to clone an unknown quantum state, i.e., given a state $|\psi\rangle$ produce a state $|\psi\rangle \otimes |\psi\rangle$.

Reconcile this theorem with (a) the copying action of the CNOT gate, and (b) quantum teleportation.

(If you are interested, see, e.g., arXiv:quant-ph/9607018v1 [`http://arxiv.org/abs/quant-ph/9607018v1`] for the presentation of an "approximate cloner.")

8. Let $\mathcal{D}(\psi)$ denote the probability distribution that results from measuring the superposition $|\psi\rangle$ in, say, the computational basis. Prove that if $\||\psi\rangle - |\phi\rangle\| \le \epsilon$, then $|\mathcal{D}(\psi) - \mathcal{D}(\phi)| \le \Delta\epsilon$.