QIC 710 Problem Set 5

1. Is the following two-qubit density matrix separable? Justify your answer.

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB} = |\psi\rangle\langle\psi|$. Let $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ and $\rho_B = \mathrm{Tr}_A(\rho_{AB})$. Prove that $H(\rho_A) = H(\rho_B)$. Then give an example of a mixed state σ_{AB} such that $H(\sigma_A) \neq H(\sigma_B)$.

3. Is the partial trace cyclic? (No.)

(a) For some $n \in \mathbf{N}$ find operators $A, B \in \mathcal{L}(C^n \otimes C^n)$ such that $\mathrm{Tr}_2(AB) \neq \mathrm{Tr}_2(BA)$. Thus even though the trace is cyclic, the *partial* trace is not.

(b) Find $X \in \mathcal{L}(\mathbf{C}^n)$ such that for all $A \in \mathcal{L}(\mathbf{C}^n)$ and $B \in \mathcal{L}(\mathbf{C}^{n^2})$ we have

$$\mathrm{Tr}_2\big((X \otimes A)B\big) = \mathrm{Tr}_2\big(B(X \otimes A)\big) \ .$$

Prove that the X you found satisfies the above equation.

4. Separable two-qubit states

Recall that a state $\rho \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \cdots)$ is *separable* (with respect to the $AB\cdots$ partition) if it can be written as a convex combination of tensor-product states, i.e.,

$$\rho = \sum_i p_i \sigma_{iA} \otimes \sigma_{iB} \otimes \cdots$$

where the $p_i \geq 0$ and $\sum_i p_i = 1$, and $\sigma_{iA}, \sigma_{iB}, \ldots$ are density matrices.

In particular, the maximally mixed state $\mathbf{1}/\big(\dim(\mathcal{H}_A)\dim(\mathcal{H}_B)\cdots\big)$ is separable. In this problem, you will prove that any state sufficiently close to the maximally mixed state is also separable (a useful fact in the context of NMR-based quantum computers). Although this is a general fact, we will specialize to the case of a state on two qubits for simplicity.

(a) Let A and B be normal $d \times d$ matrices (i.e., $AA^\dagger = A^\dagger A$, $BB^\dagger = B^\dagger B$, so they are diagonalizable). Prove that

$$|\mathrm{Tr}(AB)| \leq d\|A\|\|B\| \ .$$

(b) Let A be a positive semi-definite Hermitian matrix and B a normal matrix with eigenvalues $\{\mu_j\}$. Prove that

$$\mathrm{Tr}(AB) \geq (\min_j \mu_j)\,\mathrm{Tr}\, A \ .$$

(c) Let $\rho \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a two-qubit state such that

$$\|\rho - \frac{1}{4}\mathbf{1}\| \leq \epsilon/4 \ .$$

Prove that ρ can be expanded as

$$\rho = (1-\epsilon)\frac{1}{4}\mathbf{1} + \epsilon\sigma \ ,$$

where σ is a valid state. In particular, you need to show that

$$\sigma \doteq \frac{1}{\epsilon}(\rho - \frac{1-\epsilon}{4}\mathbf{1})$$

is positive semi-definite.

(d) Let $\sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z$. Then $\frac{1}{2}(\mathbf{1} \pm \sigma_j)$ is a projection onto one of the six single-qubit Pauli eigenstates, e.g., $\frac{1}{2}(\mathbf{1} + Z) = |0\rangle\langle 0|$.

Determine fixed scalars f and g such that for any 2×2 matrix τ,

$$\tau = \sum_{j=1}^{3} \sum_{b=0}^{1} \frac{1}{2}(\mathbf{1} + (-1)^b \sigma_j) \operatorname{Tr}\left[\tau(f\mathbf{1} + g(-1)^b \sigma_j)\right] \ . \tag{1}$$

Hint: It is convenient to use that τ can be expanded

$$\tau = s_0\mathbf{1} + \sum_{j=1}^{3} s_j \sigma_j$$

for some complex numbers $s_0, \ldots, s_3 \in \mathbf{C}$.

(e) Show that any 4×4 matrix τ can be expanded as

$$\tau = \sum_{j_1=1}^{3} \sum_{b_1=0}^{1} \sum_{j_2=1}^{3} \sum_{b_2=0}^{1} \frac{1}{2}(\mathbf{1}+(-1)^{b_1}\sigma_{j_1}) \otimes \frac{1}{2}(\mathbf{1}+(-1)^{b_2}\sigma_{j_2}) \operatorname{Tr}\left[\tau(f\mathbf{1}+g(-1)^{b_1}\sigma_{j_1}) \otimes (f\mathbf{1}+g(-1)^{b_2}\sigma_{j_2})\right] \ . \tag{2}$$

Hint: First consider the case that τ is a tensor product $\tau_1 \otimes \tau_2$, and use $\operatorname{Tr}(A \otimes B) = (\operatorname{Tr} A)(\operatorname{Tr} B)$.

(f) Now substitute into part (e) $\tau = (1-\epsilon)\frac{\mathbf{1}}{4} + \epsilon\sigma$ for a state σ, from part (c). Prove that Eq. (2) implies that τ is separable provided $\epsilon > 0$ is small enough. In particular, use the bound from part (b) to show that all coordinates in Eq. (2) are *nonnegative* for ϵ small enough. How small should ϵ be? Hint: Determine the smallest eigenvalue of $(f \pm g\sigma_j) \otimes (f \pm g\sigma_{j'})$.