Exact Corotational Linear FEM Stiffness Matrix

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Abstract

This technical report gives the exact corotational linear FEM stiffness matrix for a linear tetrahedral element. The matrix is obtained by computing the higher-order terms (corrections) originating because the element rotation varies with the tet deformation.

1 Introduction

Let $m_1, m_2, m_3, m_4 \in \mathbb{R}^3$ be the vertices of a tet in the undeformed configuration, and let $x_1, x_2, x_3, x_4 \in \mathbb{R}^3$ be their positions in the deformed configuration. The deformation gradient $F \in \mathbb{R}^{3\times 3}$ of the tet equals the upper-left 3×3 block of the 4×4 matrix PV^{-1} [Müller and Gross 2004], where

$$P = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{1}$$

Polar decomposition on F gives $F = RS$, where R is orthonormal, and S is symmetric. The corotational FEM elastic forces $f = [f_1^T, f_2^T, f_3^T, f_4^T]^T \in \mathbb{R}^{12}$ on the tet vertices equal

$$f = \hat{R} K_{\text{rest}} (\hat{R}^T x - m), \tag{2}$$

where $K_{\text{rest}} \in \mathbb{R}^{12\times 12}$ is the element stiffness matrix in the rest configuration, and we have assembled $m = [m_1^T, m_2^T, m_3^T, m_4^T]^T$ and $x = [x_1^T, x_2^T, x_3^T, x_4^T]^T$. Given any 3×3 matrix A, we will use the hat notation $\hat{A}$ to denote the 12×12 block-diagonal matrix with four diagonal 3×3 blocks A.

2 Exact tangent stiffness matrix

For implicit integration, it is necessary to compute the gradient of f with respect to x, the *tangent stiffness matrix* $K = df/dx$. This matrix is traditionally approximated [Müller and Gross 2004] as $\hat{R} K_{\text{rest}} \hat{R}^T$, where K_{rest} is the linear FEM stiffness matrix. However, because F is a function of x, so is R, and therefore the exact tangent stiffness matrix incorporates two additional terms:

$$K = \hat{R} K_{\text{rest}} \hat{R}^T + \Big[\widehat{\frac{\partial R}{\partial x^\ell}} K_{\text{rest}} \big(\hat{R}^T x - m\big)\Big]_\ell + \Big[\hat{R} K_{\text{rest}} \widehat{\frac{\partial R}{\partial x^\ell}}^T x\Big]_\ell, \tag{3}$$

where $x^\ell \in \mathbb{R}$ denotes the ℓ-th component of x, for $\ell = 1, \ldots, 12$, and $[a_\ell]_\ell$ denotes a 12×12 matrix whose ℓ-th column is $a_\ell \in \mathbb{R}^{12}$. We will now show how to compute the 3×3 matrices $\partial R/\partial x^\ell$, for $\ell = 1, \ldots, 12$. To perform the simulation, one then evaluates K using Equation 3, and uses it for implicit (backward Euler) integration.

3 Gradients of tet rotation

We compute the rotation gradients using the chain rule

$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial F} \frac{\partial F}{\partial x}. \tag{4}$$

In order to avoid the tensor notation in Equation 4, we have unrolled the entries of R and F into 9-vectors, using (say) row-major notation. Then, the first factor $\partial R/\partial F$ in Equation 4 is a 9×9 matrix, whereas the second factor $\partial F/\partial x$ and the product $\partial R/\partial x$ are 9×12 matrices. In Equation 3, we need the term $\partial R/\partial x^\ell$, which is the ℓ-th column of $\partial R/\partial x$, rolled into a 3×3 matrix.

Factor $\partial F/\partial x$ is block-sparse, and follows from Equation 1,

$$\frac{\partial F}{\partial x} = \begin{bmatrix} n_1 & 0 & 0 & n_2 & 0 & 0 & n_3 & 0 & 0 & n_4 & 0 & 0 \\ 0 & n_1 & 0 & 0 & n_2 & 0 & 0 & n_3 & 0 & 0 & n_4 & 0 \\ 0 & 0 & n_1 & 0 & 0 & n_2 & 0 & 0 & n_3 & 0 & 0 & n_4 \end{bmatrix}, \tag{5}$$

where $n_i \in \mathbb{R}^3$ contains the first three entries of the i-th row of V^{-1}.

Gradients of polar decomposition have been derived in [Barbič and Zhao 2011] and [McAdams et al. 2011], and we use them to derive $\partial R/\partial F$, as follows. For $i, j = 1, 2, 3$, let F_{ij} denote the entry of F in i-th row and j-th column. Then, we can define a mapping $\overline{F}_{ij} : \mathbb{R} \to \mathbb{R}^{3\times 3}$,

$$\overline{F}_{ij}(s) = F + s\, e_i e_j^T, \tag{6}$$

where $e_i \in \mathbb{R}^3$ is the i-th standard basis vector, $s \in \mathbb{R}$ is a scalar, and F is kept constant. For any s, polar decomposition gives $\overline{F}_{ij}(s) = R_{ij}(s) S_{ij}(s)$. Note that for $s = 0$, this decomposition has been already computed, $F = RS$; for other values of s, it is only needed as a mathematical concept. For such a 1D family of polar decompositions, it follows [Barbič and Zhao 2011]

$$\frac{\partial R}{\partial F_{ij}} = R'_{ij}(s)_{|s=0} = \widetilde{\omega_{ij}} R, \tag{7}$$

where $\omega_{ij} \in \mathbb{R}^3$ is the solution to

$$G\omega_{ij} = 2\text{skew}(R^T e_i e_j^T), \quad \text{for} \quad G = \big(\text{tr}(S)I - S\big)R^T \in \mathbb{R}^{3\times 3}. \tag{8}$$

Here, given a vector $a \in \mathbb{R}^3$, $\tilde{a}$ denotes the unique skew-symmetric matrix with the property $\tilde{a}y = a \times y$, for all $y \in \mathbb{R}^3$. For a matrix $A \in \mathbb{R}^{3\times 3}$, $\text{skew}(A) \in \mathbb{R}^3$ denotes the unique vector corresponding to its skew-symmetric part $(A - A^T)/2$, i.e., $\text{skew}(\tilde{a}) = a$. Matrix $G \in \mathbb{R}^{3\times 3}$ needs to be formed and inverted only once for each F. We then compute $\partial R/\partial F_{ij}$ using Equations 8 and 7.

Notes: Our implementation is available in Vega FEM [Barbič et al. 2012]. A similar K has been computed by [Chao et al. 2010] (tet meshes), and by [McAdams et al. 2011] (hexahedral meshes).

References

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