A Semidefinite Approach to
Information Design in Non-atomic Routing Games

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Abstract

We consider a routing game among non-atomic agents where link latency functions are conditional on an uncertain state of the network. All the agents have the same prior belief about the state, but only a fixed fraction receive private route recommendations or a common message, which are generated by a known randomization, referred to as private or public signal respectively. The remaining non-receiving agents choose route according to Bayes Nash flow with respect to the prior. We develop a computational approach to solve the optimal information design problem, i.e., to minimize expected social latency cost over all public or obedient private signals. For a fixed flow induced by non-receiving agents, design of an optimal private signal is shown to be a generalized problem of moments for affine link latency functions, and to admit an atomic solution for the basic two link case. Motivated by this, a hierarchy of polynomial optimization is proposed to approximate, with increasing accuracy, information design over private and public signals, when the non-receiving agents choose route according to Bayes Nash flow. The first level of this hierarchy is shown to be exact for the basic two link case.

I. INTRODUCTION

Route choice decision in traffic networks under uncertain and dynamic environments, such as the ones induced by recurring and unpredictable incidents, can be a daunting task for agents. Private route recommendation or public information systems could therefore play an important role in such settings. While the agents have prior about the uncertain state, e.g., through experience or publicly available historic records, the informational advantage of such systems in knowing the realization gives the possibility of inducing a range of traffic flows by selecting different route recommendation or public information strategies.

A strategy of a recommendation system to map state realization to (randomized) private route recommendations for the agents is referred to as a private signal. On the other hand, a strategy to map state realization to (randomized) public messages is referred to as a public signal. A private signal is feasible or obedient, if, to every agent, it recommends a route which is weakly better in expectation than the other routes. Under a public signal, the agents can be assumed to choose routes consistent with Bayes Nash flow with respect to the posterior. The problem of minimizing expected social latency cost over all obedient private signals or over all public signals is known as information design. In this paper, we are interested in these problems for non-atomic agents, including generalization.

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to the case when a fraction of agents do not participate in signaling and induce Bayes Nash flow with respect to the prior. The technical challenge in the general case is the joint consideration of optimal signal for receiving agents and the flow induced by non-receiving agents.

Information design for finite agents has attracted considerable attention recently with applications in multiple domains, e.g., see [1] for an overview; the single agent case was studied in [2] as Bayesian persuasion. In finite agent (and finite action) setting, the obedience condition on the signal can be expressed as finite linear constraints, one for each combination of actions by the agents. This allows to cast the information design problem as a tractable optimization problem. Techniques to further reduce computational cost of information design are presented in [3]. However, analogous computational approaches to solve information design for non-atomic agents, particularly for routing games, are lacking.

There has been a growing interest recently in understanding the impact of information in non-atomic routing games. For example, [4] demonstrates informational Braess paradox in which revealing information about all the links does not necessarily minimize social cost; [5], [6] illustrate that properly designed information structure could reduce price of anarchy; [7] demonstrates that information design only for a fraction of agents, while taking into account externality from flow induced by the rest, might be beneficial for social cost. Information design using private signals, as in this paper, has also been pursued recently in [8]. Optimal public signals for some settings were characterized in [9]. While these existing works provide useful insights, the information design aspect of these works is restricted to stylized settings involving a network with just two parallel links, deterministic signals, and link latency functions which ensure non-zero flow on all links under all state realizations. It is not apparent that the methodologies underlying these studies, which typically rely on analytical solutions, can be generalized. In this paper, on the other hand, we develop a computational approach with focus on parallel networks and affine link latency functions, as an illustration.

Our key observation is that information design has strong connections with the generalized problem of moments (GPM) [10]. A GPM minimizes, over finite probability measures, a cost which is linear in moments with respect to these measures subject to constraints which are also linear in the moments. This connection allows to leverage computational tools developed for GPM, such as GloptiPoly [11], which utilizes a hierarchy of semidefinite relaxations to lower bound GPM arbitrarily closely by relaxation of sufficiently high order, at the expense of increasing computational cost.

For a fixed flow induced by non-receiving agents, we show that information design for receiving agents is indeed a GPM. Furthermore, by exploring the specific structure of the information design problem, we show that it admits an optimal solution which is atomic for two links. Such a structural insight is useful in suggesting a natural polynomial optimization hierarchy to approximate, with increasing accuracy, information design over private signals for arbitrary number of links, while letting non-receiving agents induce Bayes Nash flow with respect to prior. The first level of this hierarchy is provably exact for two links, and the hierarchy also allows to approximate, with increasing accuracy, information design over public signals. The polynomial optimization at each level of the hierarchy in turn can be solved arbitrarily approximately using GPM and related machinery [12].
In summary, the main contributions of the paper are as follows. First, by making connection to GPM and associated semidefinite programming machinery, we point to a compelling computational framework to solve information design problems. Second, by establishing the atomicity of an optimal solution for the basic two link case, we provide some credence to such a structural assumption often implicitly made in information design studies. Third, we suggest a natural polynomial optimization hierarchy to approximate, with increasing accuracy, optimal information design over private and public signals. This hierarchy is grounded in the fact that its first level is exact for the basic case of two links. The ability of our formulation and solution methodology to handle a certain fraction of agents not participating in signaling but who induce externality on the participating agents allows to assess the value of information, an exercise which hitherto has been restricted to public signals. Overall, the computational approach proposed in this paper allows to considerably expand the scope of information design studies for non-atomic routing games, which has been limited so far to stylized settings.

The rest of the paper is organized as follows. Section II formulates the information design problem over private signals for non-atomic routing games. Section III shows the connection between this problem and GPM, and proposes a polynomial optimization hierarchy for its solution. Section IV extends the formulation and solution methodology to public signals. Section V provides illustrative simulation results, and concluding remarks are provided in Section VI.

The proofs of all the technical results are provided in the Appendix.

We end this section by defining key notations to be used throughout this paper. \( \mathbb{E}_\phi[x] \) will denote the expected value of random variable \( x \) with respect to probability distribution \( \phi \). \( \text{int}(X) \) will denote the interior of set \( X \) and \( \triangle(X) \) the set of all probability distributions on \( X \). For a vector \( x \in \mathbb{R}^n \), \( \text{diag}(x) \) will denote the \( n \times n \) diagonal matrix with elements of \( x \) on the main diagonal. For an integer \( n \), we let \( [n] := \{1, 2, \ldots, n\} \). For a vector \( x \in \mathbb{R}^n \), let \( \text{supp}(x) := \{i \in [n] \mid x_i \neq 0\} \) be the set of indices whose corresponding entries in \( x \) are not zero. For \( \lambda \geq 0 \), let \( \mathcal{P}_n(\lambda) := \left\{ x \in \mathbb{R}_{\geq 0}^n \mid \sum_{i \in [n]} x_i = \lambda \right\} \) be the \((n-1)\)-dimensional probability simplex of size \( \lambda \). \( e_{n,i} \) will be the standard \( i \)-th basis vector in \( \mathbb{R}^n \), i.e., its \( i \)-th entry is one and all the other entries are zero. \( 0_{n \times m} \) and \( 1_{n \times m} \) will denote \( n \times m \) matrices all of whose entries are 0 and 1 respectively. In all these notations, the subscripts corresponding to the size shall be omitted when clear from the context. For a matrix \( A \), its transpose is denoted as \( A^T \). For matrices \( A \) and \( B \) of the same size, their inner product is \( A \cdot B = \sum_{i,j} A_{i,j} B_{i,j} \). \( A \succeq 0 \) for a symmetric matrix \( A \) will imply that it is positive semidefinite. A principal submatrix of a square matrix \( A \) is a submatrix of \( A \) obtained by removing rows and columns from \( A \) corresponding to the same set of indices.

II. Problem Formulation and Preliminaries

Consider a network consisting of \( n \) parallel links between a single source-destination pair. Without loss of generality, let the agent population generate a unit volume of traffic demand. The latency functions on the links are affine in the flow through them, and are conditional on the state of the network \( \omega \in \Omega \):

\[
\ell_i^\omega(f_i) = \alpha_i^\omega f_i + \beta_i^\omega, \quad i \in [n]
\]

(1)

We let \( \Omega = \{\omega_1, \ldots, \omega_s\} \) be a finite set, and let \( \omega \sim \mu_0 \in \text{int}(\triangle(\Omega)) \), for some prior \( \mu_0 \) which is known to all the agents. We assume that the average value of the linear coefficient in (1) is strictly positive, i.e., \( \bar{\alpha}_i := \frac{1}{s} \sum_{\omega} \alpha_i^\omega \) is strictly positive. Let \( \mathcal{A} \) be the set of finite actions, and let \( \mathcal{X} := \{0, 1\}^n \). Let \( \mathcal{G} \) be the state space of the network, and let \( \mathcal{S}(\mathcal{G}) \) be the set of all probability distributions on \( \mathcal{G} \).
The agents do not have access to the realization of \( \omega \), but a fixed fraction \( \nu \in [0, 1] \) of the agents receives private route recommendations conditional on the realized state. These conditional recommendations are generated by a signal \( \pi = \{ \pi_\omega \in \triangle(\mathcal{P}_n(\nu)) : \omega \in \Omega \} \) as follows. Given a realization \( \omega \in \Omega \), sample a \( x \in \mathcal{P}_n(\nu) \) according to \( \pi_\omega \), and partition the agent population into \( n + 1 \) parts with volumes \( (x_1, \ldots, x_n, 1 - \nu) \). All the agents are identical, and therefore in the non-atomic setting that we are considering here the partition can be formed by independently assigning every agent to a partition with probability equal to the volume of that partition. The agents in the \( (n + 1) \)-th partition, with volume \( 1 - \nu \), do not receive any recommendation, whereas all the agents in the \( i \)-th partition, \( i \in [n] \), receive recommendation to take route \( i \).

The signal \( \pi \) and the fraction \( \nu \) is publicly known to all the agents. Therefore, it is easy to see that the (joint) posterior on \( (x, \omega) \), i.e., the proportion of agents getting different recommendations and the state of the network, formed by an agent who receives recommendation \( i \in [n] \) is:

\[
\mu^{\pi,i}(x, \omega) = \frac{x_i \pi_\omega(x) \mu_0(\omega)}{\sum_{\theta \in \Omega} \int_{p \in \mathcal{P}(\nu)} p_1 \pi_\theta(p) \, dp \, \mu_0(\theta)}
\]

and the posterior formed by an agent who does not receive a recommendation is:

\[
\mu^\emptyset(x, \omega) = \pi_\omega(x) \mu_0(\omega)
\]

Remark 1: One could consider an alternate setup where the set of agents who do not participate in the signaling scheme is pre-determined. These agents do not receive a recommendation and also do not have knowledge about \( \pi \). In this case, (3) can be replaced with \( \mu^\emptyset(x, \omega) = \frac{\mu_0(\omega)}{|\mathcal{P}(\nu)|} \) obtained by replacing \( \pi_\omega \) with the uniform distribution. The methodologies developed in this paper also extend to this alternate setting.

A signal is said to be \textit{obedient} if the recommendation received by every agent is weakly better, in expectation with respect to posterior in (2), than other routes, while the non-receiving agents induce a Bayes Nash flow with respect to their posterior in (3). Formally, a \( \pi \) is said to be obedient if there exists \( y \in \mathcal{P}_n(1 - \nu) \) such that:

\[
\sum_{\omega} \int_x \ell_i(x_i + y_i) \mu^{\pi,i}(x, \omega) \, dx \leq \sum_{\omega} \int_x \ell_j(x_j + y_j) \mu^{\pi,i}(x, \omega) \, dx, \quad i, j \in [n] \]

\[
\sum_{\omega} \int_x \ell_i(x_i + y_i) \mu^\emptyset(x, \omega) \, dx \leq \sum_{\omega} \int_x \ell_j(x_j + y_j) \mu^\emptyset(x, \omega) \, dx, \quad i \in \text{supp}(y), j \in [n]
\]

Plugging the expressions of beliefs from (2) and (3), noting that the denominators on both sides of the inequalities are the same in (4), and multiplying both sides of the second set of inequalities by \( y_j \), one gets:

\[
\sum_{\omega} \int_x \ell_i(x_i + y_i) x_i \, d\pi_\omega \mu_0(\omega) \leq \sum_{\omega} \int_x \ell_j(x_j + y_j) x_i \, d\pi_\omega \mu_0(\omega), \quad i, j \in [n]
\]

\[
\sum_{\omega} \int_x \ell_i(x_i + y_i) y_i \, d\pi_\omega \mu_0(\omega) \leq \sum_{\omega} \int_x \ell_j(x_j + y_j) y_i \, d\pi_\omega \mu_0(\omega), \quad i, j \in [n]
\]

Throughout the paper, unless noted otherwise, the summation over indices for state and link, such as \( \omega \) and \( i \), respectively, are to be taken over the entire range, i.e., \( \Omega \) and \( [n] \), respectively.
The social cost is taken to be the expected total latency:

\[ J(\pi, y) := \sum_{\omega, i} \int (x_i + y_i) \ell_i(x_i + y_i) \, d\pi_\omega \mu_0(\omega) \]  \hspace{1cm} (6)

The information design problem can then be stated as

\[ \min_{(\pi, y) \in \Pi \times \mathcal{P}(1-\nu)} J(\pi, y) \text{ s.t. (5), (7)} \]

where \( \Pi \) is the concise notation for \( \Delta(\mathcal{P}(\nu)) \).

Remark 2: The revelation principle, e.g., see [1], implies that optimality in the class of obedient direct private signals, i.e., signals which recommend routes, also ensures optimality within a broader class which includes indirect signals. An indirect signal provides noisy information about the state realization. The route choice is then determined by Bayes Nash flow with respect to the posterior beliefs induced by the signal. In Section IV, we consider a special case of indirect signals, known as public signals.

Computing solution to (7) is challenging, not the least because it involves optimizing over probability distributions. The next section proposes approximations which are provably accurate in some cases.

III. A SEMIDEFINITE APPROACH FOR PRIVATE SIGNALS

For a given \( \pi \in \Pi \), there exists a unique \( y \in \mathcal{P}(1-\nu) \) satisfying (5b), and therefore minimizing \( J(\pi, y) \) with respect to \( y \) for a fixed \( \pi \) is trivial.

**Lemma 1:** For every \( \pi \in \Pi \), there exists a unique \( y \in \mathcal{P}(1-\nu) \) satisfying (5b). Such a \( y \) is the unique solution to the following convex problem:

\[ \min_{y \in \mathcal{P}(1-\nu)} \sum_i \alpha_i y_i^2 + \left( \sum_{\omega} \mu_0(\omega) \alpha_\omega \mathbb{E}_{\pi_\omega}[x_i] + \beta_i \right) y_i \]

Lemma 1 follows from a straightforward adaptation of the standard argument for Wardrop equilibrium in the deterministic case. A short proof is provided in Section A for the sake of completeness.

We now turn our attention to minimizing \( J(\pi, y) \) over \( \pi \) satisfying (5a), for a fixed \( y \). Note that, for \( y = 0 \), this corresponds to the information design problem in the special case when \( \nu = 1 \). Even in this special case, which has been studied previously in [6], [8], no comprehensive solution methodology exists.

We start by rewriting the information design problem in terms of moments of the signal \( \pi \). Let \( z = [1, x_1, \ldots, x_n]^T \), and

\[ Z = \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \]

The information design problem for a fixed \( y \) can then be written as:

\[ \min_{\pi \in \Pi} \sum_{\omega} \int C_\omega(y) \cdot Z \, d\pi_\omega \quad (8a) \]

\[ \text{s.t. } \sum_{\omega} \int A_{(i,j)}(y) \cdot Z \, d\pi_\omega \geq 0, \quad i, j \in [n] \quad (8b) \]
where $C_\omega$ and $A_{\omega,ij}^{(k)}$ are appropriate symmetric matrices whose expressions are provided in Section 3. The cost in (8a) is the same as the cost in (6), and (8b) corresponds to the obedience constraint in (5a). GPM (8) is an instance of the generalized problem of moments (GPM) [10], which in turn can be solved numerically using GloptiPoly [11]. This software solves GPM by lower bounding it with semidefinite relaxations of increasing order. The stopping criterion on the order is however problem-dependent; approximations can be obtained by a user-specified order.

The discussion above suggests a natural alternating heuristic for solving (7): start with an arbitrary $y \in \mathcal{P}(1-\nu)$, and alternate between solving (8) for a fixed $y$ and finding a feasible $y$ using Lemma 1. Under appropriate conditions on the latency functions, one can show that this heuristic results in a sequence of feasible $(\pi, y)$ whose associated cost is monotonically decreasing, and hence convergent, though not necessarily to a global optimal value of (7).

In the next section, we propose an alternate approach for simultaneous optimization over $\pi$ and $y$. This is achieved by considering a sub-class of $\Pi$ over which (7) becomes a polynomial optimization problem. Increasing the span of this sub-class within $\Pi$ then allows to solve (7) with increasing accuracy using polynomial optimization.

A. Polynomial Optimization Hierarchy using Atomic Signals

A signal $\pi$ is called $m$-atomic, $m \in \mathbb{N}$, if, for every $\omega \in \Omega$, $\pi$ is supported on $m$ discrete points $x^{(k)} \in \mathcal{P}(\nu)$. Let the set of such signals be denoted as $\Pi(m)$. It is easy to see that every signal in $\Pi(m)$ can be represented as a $s \times m$ row stochastic matrix. To emphasize the matrix notation, we let $\pi(k|\omega)$ denote the probability of recommending routes according to $x^{(k)}$ when the state realization is $\omega$. Computing optimal signal in $\Pi(m)$ can be written as the following polynomial optimization problem:

$$\min_{x^{(k)} \in \mathcal{P}(\nu), \pi \in \Pi(m)} \sum_{k, \omega, i} \left( x_i^{(k)} + y_i \right) \ell_i(x_i^{(k)} + y_i) \pi(k|\omega) \mu_0(\omega)$$ (9a)

subject to

$$\sum_{k, \omega} \ell_i(x_i^{(k)} + y_i) x_i^{(k)} \pi(k|\omega) \mu_0(\omega) \leq \sum_{k, \omega} \ell_j(x_j^{(k)} + y_j) x_j^{(k)} \pi(k|\omega) \mu_0(\omega), \quad i, j \in [n]$$ (9b)

$$\sum_{k, \omega} \ell_i(x_i^{(k)} + y_i) y_i \pi(k|\omega) \mu_0(\omega) \leq \sum_{k, \omega} \ell_j(x_j^{(k)} + y_j) y_j \pi(k|\omega) \mu_0(\omega), \quad i, j \in [n]$$ (9c)

In particular, for the affine latency function in (1), the polynomials in the cost functions and the constraints are of degree 3. (9) can also be solved (approximately) using GloptiPoly. While (9) gives an upper bound to (7) for every $m \in \mathbb{N}$, it is natural to expect that the gap goes to zero as $m \to \infty$. However, in some special cases, (7) is equivalent to (9) for some finite $m$. The next section discusses one such case.

B. Diagonal Atomic Signal

An atomic signal which has attracted particular attention is when $\pi$ is the identity matrix of size $s$. We shall refer to such a signal as a diagonal atomic signal, and denote its finite support as $x^\omega$, $\omega \in \Omega$. The polynomial

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2 Throughout the paper, unless noted otherwise, the summation over index for discrete support, such as $k$, is to be taken over the entire range, i.e., $m$. 
optimization problem in (9) in this case simplifies to:
\[
\min_{x^\omega \in \mathcal{P}(\nu), \omega \in \Omega} \sum_{i=1}^{n} \left( x^\omega_i + y_i \right) \ell_i(x^\omega_i + y_i) \mu_0(\omega)
\]
subject to
\[
\sum_{i \neq j} \ell_i(x^\omega_i + y_i) x^\omega_j \mu_0(\omega) \leq \sum_{i \neq j} \ell_j(x^\omega_j + y_j) x^\omega_i \mu_0(\omega), \quad i, j \in [n]
\]
In general, (10) gives an upper bound to (9) for \( m \geq s \), and hence also for (7). Yet, (10) has been used as a framework for optimal design of private signals for \( \nu = 1 \), e.g., in [6], without formal justification for its equivalence to (7).

The next result establishes this equivalence in a special case, and also establishes that (10) is equivalent to the following semidefinite program:

\[
\min_{M \succeq 0} J(M) := C \cdot M
\]
subject to
\[
A^{(i,j)} \cdot M \succeq 0, \quad i, j \in [n]
\]
\[
B^{(i,j)} \cdot M \succeq 0, \quad i, j \in [n]
\]
\[
M(1,1) = 1
\]
\[
M(i,j) \geq 0, \quad i, j \in [(s+1)n + 1]
\]
\[
S_x^{(k)} \cdot M = 0, \quad S_y \cdot M = 0, \quad k \in [m]
\]
\[
T_x^{(i,k)} \cdot M = 0, \quad T_y^{(i)} \cdot M = 0, \quad i \in [n], k \in [m]
\]
where the expressions for symmetric matrices \( C, A^{(i,j)}, B^{(i,j)}, S_x^{(k)}, S_y, T_x^{(i,k)} \) and \( T_y^{(i)} \) are provided in Section B.

**Theorem 1:** If \( n = 2 \), then (10), (7) and (11) are all equivalent to each other for affine latency functions in (4).

**Remark 3:** 1) For \( n = 2 \), if \( \hat{M}^* = \begin{bmatrix} 1 & \hat{m}^* \hat{m}^*^T \\ \hat{m}^*^T & M_0^* \end{bmatrix} \) is an optimal solution to (11), then \( [x_1^{\omega_1}, x_2^{\omega_1}, \ldots, x_1^{\omega_2}, x_2^{\omega_2}, y_1, y_2]^T = \hat{m}^* \) is an optimal solution for (10), and hence also for (7). The numerical implication of the proof of Theorem 1 are that GloptiPoly solves (10) with relaxation order two. This is utilized for simulations in Section V.

2) Theorem 1 and its proof approach might appear to be generalization of an observation in [8], which was made for \( \nu = 1 \), and under constraints on the coefficients in the affine form of the link latency functions. Not only do we remove these restrictions, but more importantly, our proof implicitly highlights that the obedience constraint needs more careful treatment than suggested in [8].

**IV. PUBLIC SIGNALS**

A public signal is an indirect signal, under which, for every state realization, \( \nu \) fraction of agents all receive the same message among \( \{1, \ldots, m\} = [m] \). Formally, a public signal is a map \( \pi^{pub} : \Omega \to \Delta([m]) \), or can alternately be represented as a \( s \times m \) row stochastic matrix. The posterior formed by agents when the message they receive is \( k \) is:

\[
\mu^{\pi^{pub}, k}(\omega) = \frac{\pi^{pub}(k|\omega) \mu_0(\omega)}{\sum_{\theta} \pi^{pub}(k|\theta) \mu_0(\theta)} \quad \omega \in \Omega
\]

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The joint posterior formed by agents who do not receive message, but have knowledge of $\pi^{\text{pub}}$, is:

$$
\mu^{\text{pub}, \emptyset}(k, \omega) = \pi^{\text{pub}}(k|\omega)\mu_0(\omega), \quad k \in [m], \; \omega \in \Omega
$$

(13)

Public signals over $m$ messages have strong parallel with, but are not equivalent to, $m$-atomic private signals considered in Section III-A. We return to this connection in Remark 4.

Let $x^{(k)} \in \mathcal{P}(\nu)$ be the link flow induced by receiving agents, when the message they receive is $k \in [m]$, and let $y \in \mathcal{P}(1 - \nu)$ be the link flow induced by agents not receiving the message. $x^{(k)}$ is the Bayes Nash flow with respect to the posterior in (12) and $y$ is the Bayes Nash flow with respect to the posterior in (13). That is, $x^{(k)}$ satisfies:

$$
\sum_\omega \ell_\omega^i (x^{(k)}_i + y_i)\mu^{\text{pub}, k}(\omega) \leq \sum_\omega \ell_\omega^j (x^{(k)}_j + y_j)\mu^{\text{pub}, k}(\omega), \quad i \in \text{supp}(x^{(k)}), \; j \in [n]
$$

(14)

Substituting the expression from (12), the conditions on $\{x^{(1)}, \ldots, x^{(m)}\}$ can be collectively rewritten as

$$
x^{(k)}_i \left( L^{(k)}_i (\nu, x, y) - L^{(k)}_j (\nu, x, y) \right) \leq 0, \quad i, j \in [n], \; k \in [m]
$$

(15)

Similarly, the condition on $y$ can be written as

$$
y_i \sum_k \left( L^{(k)}_i (\nu, x, y) - L^{(k)}_j (\nu, x, y) \right) \leq 0, \quad i, j \in [n]
$$

(16)

The social cost is:

$$
J(\pi^{\text{pub}}, x, y) := \sum_k \sum_i \sum_\omega (x^{(k)}_i + y_i) \ell_\omega^i (x^{(k)}_i + y_i) \pi^{\text{pub}}(k|\omega)\mu_0(\omega)
$$

(17)

$$
= \sum_k \sum_i (x^{(k)}_i + y_i) L^{(k)}_i (\pi^{\text{pub}}, x, y)
$$

Therefore, the problem of optimal public signal design can be written as:

$$
\min_{x^{(k)} \in \mathcal{P}(\nu), \; k \in [m]} \min_{y \in \mathcal{P}(1 - \nu)} J(\pi^{\text{pub}}, x, y) \quad \text{s.t. (14) - (16)}
$$

(18)

where $\Pi^{\text{pub}}(m)$ is the set of $s \times m$ row stochastic matrices. Similar to (9), (18) is a third degree polynomial optimization problem for affine latency functions.

**Remark 4:** It is interesting to compare the formulations in (9) and (18) for $m$-atomic private signals and public signals with $m$ messages respectively. The costs in (9a) and (17) can be easily seen to be identical; and so are the constraints in (9c) and (16). (14) implies (9b) but not vice-versa in general. This is consistent with the interpretation that every public signal with $m$ messages is also an $m$-atomic private signal.
A. Fixed Public Signal

It is sometimes of interest to evaluate the cost of a given public signal. Two such signals have attracted particular interest, full information and no information:

\[
\pi_{\text{pub, full}} = \begin{bmatrix}
\omega_1 & 1 & 0 & \ldots & 0 \\
\omega_2 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\omega_s & 0 & 0 & \ldots & 1
\end{bmatrix},
\pi_{\text{pub, no}} = \begin{bmatrix}
\omega_1 & 1 & 0 & \ldots & 0 \\
\omega_2 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\omega_s & 1 & 0 & \ldots & 0
\end{bmatrix},
\]

where \( m = s \) for the full information signal, and \( m \) is arbitrary for the no information signal. The cost in (17) for any public signal can be computed once the induced flows \( x^{(k)}, k \in [m] \), and \( y \) are known. These are given by the next result.

**Lemma 2:** The link flows induced by a public signal \( \pi_{\text{pub}} \) are solutions to

\[
\min_{y \in \mathcal{P}(1-\nu)} \sum_{i,\omega,k} \int_0^{x_{k,i}+y_i} \ell_i(z) dz \pi_{\text{pub}}(k|\omega) \mu_0(\omega)
\]

The proof of Lemma 2 follows along the same lines as that of Lemma 1. (19) is convex and therefore the KKT conditions, which are necessary and sufficient, are equivalent to (14) and (16).

V. Simulations

Figure 1 compares the minimum cost achievable under private signals, public signals with \( m = 2 \) messages, and full information for affine latency functions over two parallel links. The simulation parameters are: \( n = 2, s = 2, \)

\[
\alpha = \begin{bmatrix}
i=1 & i=2 \\
\omega_1 & 4 & 2 \\
\omega_2 & 1 & 2
\end{bmatrix}, \quad \beta = \begin{bmatrix}
i=1 & i=2 \\
\omega_1 & 5 & 25 \\
\omega_2 & 20 & 15
\end{bmatrix}, \quad \mu_0 = \begin{bmatrix}
\omega_1 & 0.6 \\
\omega_2 & 0.4
\end{bmatrix}
\]

and the total demand was set to be 5.

The optimal costs under private and public signals for every \( \nu \in [0,1] \) were computed using GloptiPoly. Following Theorem 1, (18) was used for private signals, and (19) was used for public signals; a relaxation order of 3 was found to be sufficient in GloptiPoly.

While the cost shows non-monotonic behavior with respect to \( \nu \) in the full information case, the optimal cost is monotonically non-decreasing under private and public signals. Expectedly, the optimal cost under public signal is no greater than the cost under full information, and the optimal cost under private signal is no greater than under public signal. Interestingly, in this case, for small values of \( \nu \), full information is an optimal public signal, but an optimal private signal gives strictly lower cost than the full information case for all \( \nu \in (0,1) \). The no information signal corresponds to \( \nu = 0 \), when all the costs are expectedly equal. The minimum social cost for these simulation parameters is 83.33.
VI. CONCLUSION AND FUTURE WORK

Information design for non-atomic routing games is gaining increasing attention. While existing works provide useful insights through analysis of simple scenarios, the generality of these insights is not readily apparent. Relatedly, a computational approach to operationalize optimal information design for general settings does not exist to the best of our knowledge. By making connection to semidefinite programming (SDP), this paper not only fills this gap, but also allows to leverage computational tools developed by the SDP community. The latter is particularly relevant for extending the approach to general non-atomic games.

There are several immediate directions for future work, which will be added to the working draft [13]. Extending Theorem 1 to $n > 2$, and onwards to general networks is clearly relevant. Similarly, while the overall SDP based machinery extends to general polynomial latency functions, providing computational guarantees such as in Theorem 1 remains open. A practically relevant instance in this context is the BPR latency function [14]. In all these directions, deriving the counterpart of Theorem 1 for public signals is also interesting. The observations in Section V point to several interesting conjectures. One such observation is that the optimal cost is monotonically non-increasing in $\nu$ under private and public signals, even if the cost under a specific signal may not exhibit monotonicity. It also remains to investigate the gap between optimal cost under private and public signals as $m \to \infty$; this gap is zero for values of $\nu$ closer to one for the example in Section V. Finally, it would be interesting to utilize the approach in this paper to quantify the reduction in price of anarchy under information design. This will complement preliminary work, e.g., in [5], where such an analysis is provided under specific models for correlation between coefficients of affine latency functions across links, and under a specific class of signals.

REFERENCES

A. Proof of Lemma 1

Strict convexity of (1) follows from the assumption that $\pi_i > 0$ for all $i \in [n]$. Combining this with convexity of $P(1-\nu)$ implies that (1) admits a unique solution. The KKT conditions, which are necessary and sufficient in this case, state that $y \in P(1-\nu)$ is a solution if there exist $\eta_i \geq 0$ and $\gamma \in \mathbb{R}$ such that, for all $i \in [n]$,$$
abla \mu_0(\omega) \mathbb{E}_{\pi_\omega} [\alpha_i^\omega (x_i + y_i) + \beta_i^\omega] - \eta_i + \gamma = 0$$ and $\eta_i y_i = 0$. Equivalence between this and (5b) then follows standard arguments for Wardrop equilibrium.

B. Matrix Expressions

In all the matrices below, the lower triangular entries, generically represented as *, are equal to their upper triangular counterparts.

\[
C_\omega(y) = \mu_0(\omega) \begin{bmatrix}
y^T \text{diag}(\alpha^\omega) y + y^T \beta^\omega \frac{\beta^\omega}{2} + y^T \text{diag}(\alpha^\omega) \\
* 
\end{bmatrix}, \quad \alpha^\omega = [\alpha_1^\omega, \ldots, \alpha_n^\omega]^T, \quad \beta^\omega = [\beta_1^\omega, \ldots, \beta_n^\omega]^T
\]

\[
A^{(i,j)}_\omega(y) = \begin{bmatrix}
0 \\
\mu_0(\omega) \frac{\alpha_i^\omega y_j - \alpha_j^\omega y_i + \beta_i^\omega - \beta_j^\omega}{2} e_i^T
\end{bmatrix}, \quad \tilde{A}^{(i,j)}_\omega = \mu_0(\omega) \left( \frac{\alpha_i^\omega}{2} e_i e_j^T - \frac{\alpha_j^\omega}{2} e_j e_i^T \right)
\]
\[
C = \begin{bmatrix}
\begin{array}{cccccc}
& x^ω_1 & & & & \\
0 & \frac{μ_0(ω_1)}{2}β_ω T & \cdots & \frac{μ_0(ω_s)}{2}β_ω T & \cdots & \frac{y}{2} \\
& \ast & μ_0(ω_1)\text{diag}(αω_1) & \cdots & 0 & μ_0(ω_s)\text{diag}(αω_s) \\
& \vdots & \vdots & \cdots & \vdots & \vdots \\
& \ast & \ast & \cdots & μ_0(ω_s)\text{diag}(αω_s) & μ_0(ω_s)\text{diag}(αω_s) \\
y & \ast & \ast & \cdots & \ast & \text{diag}(π)
\end{array}
\end{bmatrix}
\]

\[
A^{(i,j)} = \begin{bmatrix}
\begin{array}{cccccc}
& x^ω_1 & & & & \\
0 & μ_0(ω_1) & \ast & \cdots & \ast & \ast \\
& μ_0(ω_1) & A^{ω_1}_ω & \ast & \cdots & \ast & \ast \\
& \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
& \ast & \ast & \cdots & \ast & \ast \\
y & \ast & \ast & \cdots & \ast & \ast
\end{array}
\end{bmatrix}
\]

\[
B^{(i,j)} = \begin{bmatrix}
\begin{array}{cccccc}
& x^ω_1 & & & & \\
0 & 0 & \ast & \cdots & \ast & \ast \\
& μ_0(ω_1) & A^{ω_1}_ω & \ast & \cdots & \ast & \ast \\
& \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
& \ast & \ast & \cdots & \ast & \ast \\
y & \ast & \ast & \cdots & \ast & \ast
\end{array}
\end{bmatrix}
\]

\[
S_x^{(k)} = x^ω_x
\]

\[
S_y = \begin{bmatrix}
\begin{array}{cccccc}
& x^ω_1 & & & & \\
ν & 0 & \ast & \cdots & \ast & \ast \\
& 0 & 0 & \ast & \cdots & \ast & \ast \\
& \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
& \ast & \ast & \cdots & \ast & \ast \\
y & \ast & \ast & \cdots & \ast & \ast
\end{array}
\end{bmatrix}
\]

\[

\begin{bmatrix}
\begin{array}{cccccc}
-ν & 0 & \ast & \cdots & \ast & \ast \\
0 & 0 & \ast & \cdots & \ast & \ast \\
& \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
1/2 & 0 & \ast & \cdots & \ast & \ast \\
& \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \ast & \cdots & \ast & \ast \\
y & 0 & 0 & \ast & \cdots & \ast \\
\end{array}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\begin{array}{cccccc}
ν & 0 & \ast & \cdots & \ast & \ast \\
0 & 0 & \ast & \cdots & \ast & \ast \\
& \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \ast & \cdots & \ast & \ast \\
y & 0 & 0 & \ast & \cdots & \ast \\
\end{array}
\end{bmatrix}
\]

\[

\begin{bmatrix}
\begin{array}{cccccc}
ν & 0 & \ast & \cdots & \ast & \ast \\
0 & 0 & \ast & \cdots & \ast & \ast \\
& \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \ast & \cdots & \ast & \ast \\
y & 0 & 0 & \ast & \cdots & \ast \\
\end{array}
\end{bmatrix}
\]
C. Proof of Theorem \[7\]

Equivalence between \[10\] and \[7\]: Let \((\pi^*, y^*)\) be an optimal solution to \[7\]. We show that, for every \(y \in \mathcal{P}(1-\nu)\), there exists an optimal solution to \[8\] which is diagonal atomic. When specialized to \(y^*\), this establishes the equivalence.

It is sufficient to show that for every feasible \(\pi = \{\pi_\omega : \omega \in \Omega\}\), the diagonal atomic \(\pi^a := \{\pi^a_\omega(x) = \delta(x - E_\pi(x)) : \omega \in \Omega\}\) is also feasible and satisfies \(J(\pi^a) \leq J(\pi)\).

It is easy to see that \(J(\pi) - J(\pi^a) = \sum_{\omega} \mu_\omega(\omega) \text{diag}(\alpha^\omega) \cdot (\int xx^T d\pi_\omega - \int xx^T d\pi^a_\omega)\), where \(\int xx^T d\pi_\omega - \int xx^T d\pi^a_\omega\) is the covariance matrix of \(\pi_\omega\) and hence is positive semidefinite. \(\text{diag}(\alpha^\omega)\) is trivially positive semidefinite. Therefore, since the inner product of positive semidefinite matrices is non-negative, \(J(\pi) - J(\pi^a) \geq 0\).

\(\text{(8b)}\) for \(\pi, i = 1\) and \(j = 2\) is:

\[
\sum_{\omega} \int \left( \alpha^\omega_1 x_1^2 - \alpha^\omega_2 x_2 x_1 + \alpha^\omega_3 x_1 y_1 - \alpha^\omega_4 x_1 y_2 + \beta^\omega_1 x_1 - \beta^\omega_2 x_1 \right) d\pi_\omega \mu_\omega(\omega) \leq 0
\]

Plugging \(x_2 = \nu - x_1\), this is equivalent to:

\[
\sum_{\omega} \left( (\alpha^\omega_1 + \alpha^\omega_2) \int x_1^2 d\pi_\omega + (\alpha^\omega_1 y_1 + \beta^\omega_1 - \nu \alpha^\omega_2 - y_2 \alpha^\omega_2 - \beta^\omega_2) \int x_1 d\pi_\omega \right) \mu_\omega(\omega) \leq 0
\]

\[\int x_1 d\pi_\omega = \int x_1 d\pi^a_\omega\] by definition, and \(\int x_1^2 d\pi_\omega \geq (\int x_1 d\pi_\omega)^2 = \int x_1^2 d\pi^a_\omega\) by Jensen’s inequality. Therefore,

\[
\sum_{\omega} \left( (\alpha^\omega_1 + \alpha^\omega_2) \int x_1^2 d\pi^a_\omega + (\alpha^\omega_1 y_1 + \beta^\omega_1 - \nu \alpha^\omega_2 - y_2 \alpha^\omega_2 - \beta^\omega_2) \int x_1 d\pi^a_\omega \right) \mu_\omega(\omega) \leq 0
\]

which is equivalent to \(\text{(8b)}\) for \(\pi^a, i = 1\) and \(j = 2\). The proof for \(i = 2\) and \(j = 1\) is identical.

Equivalence between \(10\) and \(11\): \(10\) is equivalent to:

\[
\min_{\hat{\pi}} \int C \cdot \hat{Z} d\hat{\pi}
\]

s.t. \(\int A(i,j) \cdot \hat{Z} d\hat{\pi} \geq 0, \quad i, j \in [n]\)

\(\int B(i,j) \cdot \hat{Z} d\hat{\pi} \geq 0, \quad i, j \in [n]\)

\(\hat{\pi}\) is 1-atomic

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where the expressions for the symmetric matrices $C$, $A^{(i,j)}$, and $B^{(i,j)}$ are in Section 8.

\[
\tilde{Z} = \begin{bmatrix} 1 & \hat{\hat{z}}^T \\ \hat{\hat{z}} & \hat{\hat{z}} \hat{\hat{z}}^T \end{bmatrix}, \quad \hat{\hat{z}} = [x_1^{\omega_1}, \ldots, x_n^{\omega_1}, \ldots, x_1^{\omega_s}, \ldots, x_n^{\omega_s}, y_1, \ldots, y_n]^T
\]

and $\tilde{\Pi}$ is the set of probability distributions over $\hat{\hat{z}}$ satisfying $x^{\omega_k} \in \mathcal{P}(\nu)$ for all $k \in [s]$ and $y \in \mathcal{P}(1 - \nu)$.

It therefore suffices to establish the equivalence between (20) and (11). We do this via a constrained version of (11):

\[
\min_{M \succeq 0} \hat{J}(M) \quad \text{s.t.} \quad (11b) - (11e), \quad \text{rank}(M) = 1 \tag{21}
\]

Specifically, (a) for every $\hat{\pi}$ feasible for (20), $M(\hat{\pi}) := \int \tilde{Z} \, d\hat{\pi}$ is feasible for (21), and hence also for (11); (b) for every $M = \begin{bmatrix} 1 & \hat{m}^T \\ \hat{m} & M^0 \end{bmatrix}$ feasible for (21), $\hat{\pi} = \delta(\hat{\hat{z}} - \hat{m})$ is feasible for (20); and (c) there exists an optimal solution $M^*$ for (11) such that $\text{rank}(M^*) = 1$. (a) and (b) together imply the equivalence between (20) and (21), and (c) implies the equivalence between (21) and (11). The proofs for these claims are as follows.

(a) For a 1-atomic $\hat{\pi}$, $M(\hat{\pi}) = \begin{bmatrix} 1 & \hat{m}^T \\ \hat{m} & M^0 \end{bmatrix}$ implying that $M(\hat{\pi})$ is rank one and positive semidefinite. $M(\hat{\pi})$ satisfying (11b) and (11c) follow from the corresponding constraints in (20). (11d) follows from the definition of $M(\hat{\pi})$, and the rest of the constraints in (11) follow from constraints on the support of $\hat{\pi}$.

(b) Proposition 1 implies that the 1-atomic $\hat{\pi} = \delta(\hat{\hat{z}} - \hat{m})$ belongs to $\tilde{\Pi}$. Simple algebra shows the equivalence between the other constraints in (20) and the corresponding constraints in (11).

(c) It is sufficient to show that, for every $M = \begin{bmatrix} 1 & \hat{m}^T \\ \hat{m} & M^0 \end{bmatrix}$ feasible for (11), the rank one $\hat{M} = \begin{bmatrix} 1 & \hat{m}^T \\ \hat{m} & \hat{m} \hat{m}^T \end{bmatrix}$ is also feasible and satisfies $\hat{J}(M) \geq \hat{J}(\hat{M})$.

$\hat{J}(M) - \hat{J}(\hat{M}) = C^0 \cdot (M^0 - \hat{m} \hat{m}^T)$, where $C^0$ is the principal submatrix of $C$ obtained by removing the first row and the first column. $M \succeq 0$ implies $M^0 - \hat{m} \hat{m}^T \succeq 0$. It is easy to see that $C^0$ is positive semidefinite.

Since the inner product of positive semidefinite matrices is non-negative, this implies that $\hat{J}(M) - \hat{J}(\hat{M}) \geq 0$. Feasibility of (11b) - (11e) follows from the definition of $\hat{M}$. It is easy to see that $S_x^{(k)} \cdot \hat{M} = S_x^{(k)} \cdot M$ and $S_y \cdot \hat{M} = S \cdot M$, and therefore (11f) is also satisfied. Also, for all $i \in [n]$ and $k \in [s]$,

\[
T_x^{(i,k)} \cdot \hat{M} = -\nu \hat{m}_i + (1 - \nu) \cdot (\hat{m} \hat{m}^T) = -\nu \hat{m}_i + \sum_j \hat{m}_i \hat{m}_j = -\nu \hat{m}_i + \nu \hat{m}_i = 0
\]

implying $\hat{M}$. Similarly, $T_y^{(i)} \cdot \hat{M} = 0$ for all $i \in [n]$, implying (11g).

(11b) for $M$ for $i = 1$ and $j = 2$ is:

\[
\sum_k \left( \alpha_1^{\omega_k} M_{2(k-1)+1,2(k-1)+1}^0 - \alpha_2^{\omega_k} M_{2(k-1)+1,2k}^0 + \alpha_1^{\omega_k} M_{2(k-1)+1,2s+1}^0 - \alpha_2^{\omega_k} M_{2(k-1)+1,2s+2}^0 \right. \\
+ \left( \beta_1^{\omega_k} - \beta_2^{\omega_k} \right) \hat{m}_{2(k-1)+1} \cdot \mu_0(\omega_k) \leq 0
\]

Plugging $M_{2(k-1)+1,2k}^0 = \nu \hat{m}_{2(k-1)+1} - M_{2(k-1)+1,2(k-1)+1}^0$ and $M_{2(k-1)+1,2s+2}^0 = (1 - \nu) \hat{m}_{2(k-1)+1}$
We are also specifically interested in probability measures over the set of all \(D\). Technical Results

\[ M_{2(k-1)+1,2s+1}, \text{ this is equivalent to} \]
\[
\sum_k (\alpha_1^{\omega_k} + \alpha_2^{\omega_k})(M_{2(k-1)+1,2(k-1)+1}^0 + M_{2(k-1)+1,2s+1}^0) + (\beta_1^{\omega_k} - \beta_2^{\omega_k} - \alpha_2^{\omega_k})\hat{m}_{2(k-1)+1}\mu_0(\omega_k) \leq 0
\]  
(22)

\[
M \geq 0 \implies \begin{bmatrix} 1 & \hat{m}_{2(k-1)+1} & \hat{m}_{2s+1} \\ * & M_{2(k-1)+1,2(k-1)+1}^0 & M_{2(k-1)+1,2s+1}^0 \\ * & * & M_{2s+1,2s+1}^0 \end{bmatrix} \succeq 0
\]

\[
\implies \begin{bmatrix} M_{2(k-1)+1,2(k-1)+1}^0 & M_{2(k-1)+1,2s+1}^0 \\ * & M_{2s+1,2s+1}^0 \end{bmatrix} - \begin{bmatrix} \hat{m}_{2(k-1)+1} \\ \hat{m}_{2s+1} \end{bmatrix} \begin{bmatrix} \hat{m}_{2(k-1)+1} & \hat{m}_{2s+1} \end{bmatrix} \succeq 0
\]

Inner product with
\[
\begin{bmatrix} \alpha_1^{\omega_k} + \alpha_2^{\omega_k} & \alpha_1^{\omega_k} + \alpha_2^{\omega_k} \\ 0 & 0 \end{bmatrix} \succeq 0 \text{ gives}
\]
\[
(\alpha_1^{\omega_k} + \alpha_2^{\omega_k})(M_{2(k-1)+1,2(k-1)+1}^0 + M_{2(k-1)+1,2s+1}^0) \geq (\alpha_1^{\omega_k} + \alpha_2^{\omega_k})(\hat{m}_{2(k-1)+1}^2 + \hat{m}_{2(k-1)+1}\hat{m}_{2s+1})
\]

Plugging into (22) implies that (11b) is satisfied by \(\hat{M}\) for \(i = 1\) and \(j = 2\). The proof for \(i = 2\) and \(j = 1\), as well as for (11c), follows similarly.

\[ D. \text{ Technical Results} \]

\[ \text{Lemma 3: Consider a real symmetric matrix } X \text{ with rank } r. X \text{ is positive semidefinite if and only if it has a positive semidefinite principal submatrix of rank } r. \]

\[ \text{Proof: The necessary condition is obvious.} \]

Without loss of generality, let the principal submatrix, say \(\tilde{X}\), of rank \(r\) be the leading one of order \(r\). The leading principal submatrix of order \(r + 1\) is singular since its rank is \(r\). Its leading principal minors are also leading principal minors of \(\tilde{X}\), and therefore nonnegative. This implies that the leading principal submatrix of order \(r + 1\) is positive semidefinite. Induction then establishes the sufficient condition.

We need additional definitions for the next result. These are adapted from [15]. A truncated moment sequence (tms) in \(n\) variables and of degree \(d\) is a finite sequence \(t = (t_a)\) indexed by nonnegative integer vectors \(a := (a_1, \ldots, a_n) \in \mathbb{N}^n\) with \(|a| := a_1 + \ldots + a_n \leq d\). Given a set \(K\), a tms \(t\) is said to admit a \(K\)-probability measure \(\zeta\), i.e., a nonnegative Borel measure supported in \(K\) with \(\int_K d\zeta = 1\), if
\[
t_a = \int_K x^a d\zeta, \quad \forall a \in \mathbb{N}^n : |a| \leq d
\]

where \(x^a = x_1^{a_1} \ldots x_n^{a_n}\) for \(x = (x_1, \ldots, x_n)\) and \(a = (a_1, \ldots, a_n)\).

We are interested in tms of degree 2. Accordingly, for brevity in notation, let
\[
t_i := t_{(0, \ldots, 0, 1, 0, \ldots, 0)}, \quad t_{i,j} := t_{(0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)}, \quad i, j \in [\tilde{n}]
\]  
(23)

We are also specifically interested in probability measures over the set of all \(\tilde{z}\) in \(\mathbb{R}^\tilde{n}\) satisfying \(\sum_{i \in [(k-1)n+1:kn]} \tilde{z}_i = \nu\) for all \(k \in [s]\) and \(\sum_{i \in [sn+1:(s+1)n]} \tilde{z}_i = 1 - \nu\). Let this set be denoted as \(\mathbb{P}(\nu)\).
Proposition 1: If a tms $t$ in $\hat{n} = n(s + 1)$ variables and of degree 2 satisfies:

$$M(t) := \begin{bmatrix} 1 & t_1 & \ldots & t_{\hat{n}} \\ t_1 & t_{1,1} & \ldots & t_{1,\hat{n}} \\ \vdots & \vdots & \ddots & \vdots \\ t_{\hat{n}} & t_{\hat{n},1} & \ldots & t_{\hat{n},\hat{n}} \end{bmatrix} \geq 0; \quad t_i \geq 0, \ i \in [\hat{n}]; \quad t_{i,j} \geq 0, \ i,j \in [\hat{n}]$$

(24a)

$$\sum_{i \in [(k-1)n+1:kn]} t_i = \nu, \ k \in [s]; \quad \sum_{j \in [(k-1)n+1:kn]} t_{i,j} = \nu t_i, \ i \in [(k-1)n+1: \hat{n}], \ k \in [s]$$

(24b)

$$\sum_{i \in [sn+1:(s+1)n]} t_i = 1 - \nu; \quad \sum_{j \in [sn+1:(s+1)n]} t_{i,j} = (1 - \nu)t_i, \ i \in [sn+1:(s+1)n]$$

$$\text{rank}(M(t)) = 1$$

then it admits a unique $\mathbb{P}(\nu)$-probability measure, which is also 1-atomic and given by $\zeta(x) = \delta(x - [t_1, \ldots, t_{\hat{n}}])$.

Proof: (24b) implies that

$$t_{i,j} = t_i t_j, \quad i,j \in [\hat{n}]$$

(25)

[15, Theorem 1.1], which in turn is from [16], implies that a $t$ satisfying (24a) admits a unique $\mathbb{P}(\nu)$-probability measure if there exists a tms $w$ in $\hat{n}$ variables and of degree 4 such that it satisfies $w_a = t_a$ for all $|a| \leq 2$, and the following:

$$M(w) := \begin{bmatrix} 1 & w_1 & \ldots & w_{\hat{n}} & w_{1,1} & \ldots & w_{1,\hat{n}} & \ldots & w_{\hat{n},1} & \ldots & w_{\hat{n},\hat{n}} \\ w_1 & w_{1,1} & \ldots & w_{1,\hat{n}} & w_{1,1,1} & \ldots & w_{1,1,\hat{n}} & \ldots & w_{1,\hat{n},1} & \ldots & w_{1,\hat{n},\hat{n}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ w_{\hat{n}} & w_{\hat{n},1} & \ldots & w_{\hat{n},\hat{n}} & w_{\hat{n},1,1} & \ldots & w_{\hat{n},1,\hat{n}} & \ldots & w_{\hat{n},\hat{n},1} & \ldots & w_{\hat{n},\hat{n},\hat{n}} \end{bmatrix} \geq 0$$

$$M_i(w) := \begin{bmatrix} w_i & w_{i,1} & \ldots & w_{i,\hat{n}} \\ w_{i,1} & w_{i,1,1} & \ldots & w_{i,1,\hat{n}} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i,\hat{n}} & w_{i,\hat{n},1} & \ldots & w_{i,\hat{n},\hat{n}} \end{bmatrix} \geq 0, \ i \in [\hat{n}]$$

$$\sum_{k \in [(l-1)n+1:ln]} w_{i,j,k} = \nu w_{i,j}, \quad \sum_{k \in [sn+1:(s+1)n]} w_{i,j,k} = (1 - \nu) w_{i,j}, \quad i,j \in [\hat{n}], l \in [s]$$

(26)
where $w_i, w_{i,j}, w_{i,j,k},$ and $w_{i,j,k,l}$ are defined similar to (23). Let

$$w_{i,j,k} = t_it_jt_k, \quad w_{i,j,k,l} = t_it_jt_kt_l,$$ \hspace{1cm} $i, j, k, l \in [\tilde{n}]$ \hspace{1cm} (27) \hspace{1cm}

Equations (25) and (27) imply $w_{i,j,k} = t_itjt_k = w_{i,j}t_k,$ and therefore, $\sum_{k \in [(t-1)n+1:tn]} w_{i,j,k} = w_{i,j} \sum_{k \in [(t-1)n+1:tn]} t_k = \nu w_{i,j}$ for all $l \in [s]$ and $\sum_{k \in [sn+1:(s+1)n]} w_{i,j,k} = w_{i,j} \sum_{k \in [sn+1:(s+1)n]} t_k = (1 - \nu)w_{i,j}$. (27) implies that every column of $M_i(w)$ is a multiple of the first column, and therefore $\text{rank}(M_i(w)) = 1$. Since the leading entry $w_i$ is nonnegative, Lemma 3 implies that $M_i(w)$ is positive semidefinite. Along the same lines, $M(w)$ has rank one and is positive semidefinite.

Since $\text{rank}(M(w)) = 1 = \text{rank}(M(t))$, [15, Theorem 1.1] implies that the unique probability measure $\zeta$ is 1-atomic. The expression for $\zeta$ is then trivial from the fact that $E_\zeta[x] = [t_1, \ldots, t_\tilde{n}]^T$. 

\[\blacksquare\]