

Due Th 09/20/22 (9am; on DEN Webpage)

1. Problem 2.2 from the Hespanha's book (page 20; attached).
2. Consider the unforced mass-spring system

$$m \ddot{y} + g(y) = 0$$

with three different models for the spring force

- **hardening spring:** $g(y) = k(1 + y^2)y$;
- **softening spring:** $g(y) = k(1 - y^2)y$;
- **linear spring:** $g(y) = ky$,

and $k > 0$.

- (a) Determine a state-space representation of this system.
- (b) Find equilibrium points of the above systems. Discuss your observations for three different spring force models.
- (c) Is this system
 - causal,
 - time-varying,
 - linear,
 - memoryless,
 - finite-dimensional?

Explain.

- (d) For three different spring force models with $m = k = 1$, use Matlab to simulate systems' responses from different initial conditions. Plot corresponding results in the phase plane (horizontal axis determined by position $y(t)$, vertical axis determined by velocity $\dot{y}(t)$) and discuss your observations.
3. Consider the scalar differential equation

$$\dot{x} = x(x^2 - 1).$$

- (a) Determine the equilibrium points of the above system.
- (b) Determine analytical solutions of the linearized systems that are obtained by linearizing the above system around equilibrium points that you found in (a).
- (c) Use Matlab to simulate systems' responses from different initial conditions and provide plots that illustrate how x changes with time. Use your results obtained in (b) to illustrate responses of linearized systems and compare them to the responses of nonlinear system from the same initial conditions.
- (d) The above system can be interpreted as the gradient descent dynamics that can be used to compute the solution of the following unconstrained optimization problem

$$\underset{x}{\text{minimize}} \quad g(x)$$

where $g(x) = x^2/2 - x^4/4$. Indeed, it is easy to verify that the above system can be written as

$$\dot{x} = -\frac{dg(x)}{dx}$$

Explain your computational observations in (c) using this insight.

4. Consider the unconstrained optimization problem

$$\underset{x}{\text{minimize}} \quad g(x) \tag{1}$$

where $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by

$$g(x) := \frac{1}{4} \|xx^T - \Lambda\|_F^2$$

and $\Lambda = \Lambda^T$ is a symmetric matrix. For any matrix M , the Frobenius norm $\|M\|_F$ is given by

$$\|M\|_F^2 = \text{trace}(M^T M) = \text{trace}(M M^T) = \sum_{i,j} M_{ij}^2.$$

If the matrix Λ is positive definite, i.e., if all of its eigenvalues are positive, then the optimal solution x^* to problem (1) corresponds the best rank-1 approximation to Λ which is given by

$$\hat{\Lambda} := x^*(x^*)^T \approx \Lambda. \tag{2}$$

Suppose that the matrix Λ is given by

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_1 > \lambda_2 > 0$.

- Form the gradient-flow dynamics associated with problem (1), i.e.,

$$\dot{x} = -\nabla g(x). \tag{3}$$

- Find all the equilibrium points of system (3).
- Linearize (3) around each of its equilibrium points.

Hint: There are five equilibrium points. This is a 2-dimensional version of a problem discussed in class.

Does such an equilibrium point always exist?

- (d) Assume that $b = 1/2$ and $mg\ell = 1/4$. Compute the torque $T(t)$ needed for the pendulum to fall from $\theta(0) = 0$ with constant velocity $\dot{\theta}(t) = 1, \forall t \geq 0$. Linearize the system around this trajectory. \square

2.2 (Local linearization around a trajectory). A single-wheel cart (unicycle) moving on the plane with linear velocity v and angular velocity ω can be modeled by the nonlinear system

$$\dot{p}_x = v \cos \theta, \quad \dot{p}_y = v \sin \theta, \quad \dot{\theta} = \omega, \quad (2.11)$$

where (p_x, p_y) denote the Cartesian coordinates of the wheel and θ its orientation. Regard this as a system with input $u := [v \ \omega]' \in \mathbb{R}^2$.

- (a) Construct a state-space model for this system with state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} p_x \cos \theta + (p_y - 1) \sin \theta \\ -p_x \sin \theta + (p_y - 1) \cos \theta \\ \theta \end{bmatrix}$$

and output $y := [x_1 \ x_2]' \in \mathbb{R}^2$.

- (b) Compute a local linearization for this system around the equilibrium point $x^{\text{eq}} = 0, u^{\text{eq}} = 0$.
- (c) Show that $\omega(t) = v(t) = 1, p_x(t) = \sin t, p_y(t) = 1 - \cos t, \theta(t) = t, \forall t \geq 0$ is a solution to the system.
- (d) Show that a local linearization of the system around this trajectory results in an LTI system. \square

2.3 (Feedback linearization controller). Consider the inverted pendulum in Figure 2.6.

- (a) Assume that you can directly control the system in torque, i.e., that the control input is $u = T$.

Design a feedback linearization controller to drive the pendulum to the upright position. Use the following values for the parameters: $\ell = 1 \text{ m}$, $m = 1 \text{ kg}$, $b = 0.1 \text{ N m}^{-1} \text{ s}^{-1}$, and $g = 9.8 \text{ m s}^{-2}$. Verify the performance of your system in the presence of measurement noise using Simulink[®].

- (b) Assume now that the pendulum is mounted on a cart and that you can control the cart's jerk, which is the derivative of its acceleration a . In this case,

$$T = -m \ell a \cos \theta, \quad \dot{a} = u.$$

Design a feedback linearization controller for the new system.

What happens around $\theta = \pm\pi/2$?

Note that, unfortunately, the pendulum needs to pass by one of these points for a swing-up, i.e., the motion from $\theta = \pi$ (pendulum down) to $\theta = 0$ (pendulum upright). \square

Attention! Writing the system in the carefully chosen coordinates x_1, x_2, x_3 is crucial to getting an LTI linearization. If one tried to linearize this system in the original coordinates p_x, p_y, θ with dynamics given by (2.11), one would get an LTV system.