Due Thursday 10/06/22 (9am; on DEN Webpage)

1. (a) Suppose that $A$ and $B$ are constant square matrices. Show that the state transition matrix for the time-varying system described by

$$
\dot{x}(t)=e^{-A t} B e^{A t} x(t)
$$

is

$$
\Phi(t, s)=e^{-A t} e^{(A+B)(t-s)} e^{A s}
$$

(b) If $A$ is an $n \times n$ matrix of full rank, show using the definition of the matrix exponential that

$$
\int_{0}^{t} e^{A \sigma} d \sigma=\left[e^{A t}-I\right] A^{-1}
$$

Using this result, obtain the solution to the linear time-invariant equation

$$
\dot{x}=A x+B \bar{u}, \quad x(0)=x_{0}
$$

where $\bar{u}$ is a constant $r$-dimensional vector and $B$ is an $(n \times r)$-dimensional matrix.
2. Consider the discrete-time system

$$
\begin{aligned}
x_{k+1} & =A x_{k}+B u_{k} \\
y_{k} & =C x_{k}+D u_{k} \\
x\left(k_{0}\right) & =x_{0}
\end{aligned}
$$

with constant matrices $A, B, C$, and $D$.
(a) Prove that this system is linear and time-invariant.
(b) Using the definition of the $\mathcal{Z}$-transform prove that $\mathcal{Z}\left(A^{k}\right)=z R(z)$, where $R(z):=(z I-A)^{-1}$ is the resolvent of the matrix $A$.
(c) For

$$
A=\left[\begin{array}{cc}
0 & 1 \\
-0.5 & 0.3
\end{array}\right]
$$

determine $R(z)$. From the resulting expression for the resolvent compute the state transition matrix of the above system at $k=9$.
3. Use the (matrix) exponential series to evaluate $\mathrm{e}^{A t}$ for:
(a) $A=\left[\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right]$;
(b) $A=\left[\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right]$.

Also, determine the eigenvalue decomposition (i.e., the eigenvalues and eigenvectors) of these two matrices.

Note: You are not allowed to use Matlab in this exercise.
4. Suppose $A(t)$ is an $n \times n$ time-varying matrix with continuous entries that satisfies

$$
A(t)\left(\int_{t_{0}}^{t} A(\sigma) \mathrm{d} \sigma\right)=\left(\int_{t_{0}}^{t} A(\sigma) \mathrm{d} \sigma\right) A(t)
$$

Show that the state-transition matrix $\Phi\left(t_{1}, t_{0}\right)$ can be computed as

$$
\Phi\left(t, t_{0}\right)=\exp \left(\int_{t_{0}}^{t} A(\sigma) \mathrm{d} \sigma\right)
$$

5. For the Mathieu equation,

$$
\ddot{y}(t)+(\omega-\alpha \cos (2 t)) y(t)=0
$$

use Matlab to compute the state-transition matrix with $x_{1}=y, x_{2}=\dot{y}, \omega=2, \alpha=1$. You should do your computations on the time interval of length equal to three periods of oscillations for $t_{0}=0$ and $t_{0}=1$.
6. Consider the following system

$$
\begin{align*}
{\left[\begin{array}{l}
\dot{x}_{1} \\
\dot{x}_{2}
\end{array}\right] } & =\left[\begin{array}{cc}
A_{11} & 0 \\
\alpha A_{21} & A_{22}
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right] \\
y & =\left[\begin{array}{ll}
0 & I
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right] \tag{S}
\end{align*}
$$

where $x_{1} \in \mathbb{R}^{n_{1}}$ and $x_{2} \in \mathbb{R}^{n_{2}}$ are the states, $y \in \mathbb{R}^{n_{2}}$ is the output, and $\alpha$ is the positive parameter.
(a) Determine if the matrix $A$ is normal.
(b) Find the expression for the state-transition matrix and the resolvent of the above system. The resulting state transition matrix should be partitioned conformably with the partition of the matrix $A$ and its components should be expressed in terms of $A_{11}, A_{21}, A_{22}$, and $\alpha$.
(c) For

$$
A_{11}=\left[\begin{array}{ll}
-1 & -2 \\
-2 & -5
\end{array}\right], \quad A_{22}=-1, \quad A_{21}=\left[\begin{array}{ll}
1 & 2
\end{array}\right]
$$

find the right and the the left eigenvectors of the matrix $A$. You result should be expressed in terms of the parameter $\alpha$ and the left eigenvectors should be normalized to satisfy bi-orthogonality condition.
(d) For the system given in part (c), determine the expression for the system's output arising from the initial conditions in $x_{1}$ and $x_{2}$. You should use the results obtained in part (c) here.
(e) Sketch the output components determined in part (d). How do your results change if $\alpha$ is increased? Explain your observations.

