Due Tuesday 11/22/22 (9am; on DEN Webpage)

1. Consider the system parameterized by the scalars k and R

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -(1+k^2)/R & 0 \\ k & -(2+k^2)/R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= x_2. \end{aligned}$$

   (a) For what values of k and R is this system stable?
   (b) Derive the formula for the H_2 norm of this system as a function of k and R. Using this formula, plot the H_2 norm as a function of k for $R = 1$ and $R = 1000$, and as a function of R for $k = 2$.
   (c) Find the solution of the unforced system (i.e. determine operator $G(t)$ that maps initial conditions to the output $y(t)$, $y(t) = G(t)x_0$).
   (d) Plot the maximal singular value of $G(t)$ as a function of time (on time interval $t \in (0, 1000)$) for two different cases: a) $R = 1000$, $k = 0$; b) $R = 1000$, $k = 2$. How do these two cases compare to each other. Explain the obtained results.

2. Write a MATLAB program to compute the H_∞ norm of a SISO transfer function using the grid (in frequency) method. Test your program on the function

$$H(s) = \frac{1}{s^2 + 10^{-6}s + 1},$$

   and compare your answer to the exact solution (computed by hand using the definition of the H_∞ norm).

3. Consider the following pair of matrices

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}.$$

   (a) Suppose λ_1, λ_2, and λ_3 are distinct, what are the conditions on γ_1, γ_2, and γ_3 for the pair (A, B) to be controllable?
   (b) For a discrete-time system with the above given matrices A and B, $\{\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3\}$, and $\gamma_1 = \gamma_2 = \gamma_3 = 1$, plot the energy (as a function of time for $k \geq 3$) of the minimum energy control necessary to reach the final state $x_f = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Comment your results.
   (c) Suppose λ_1, λ_2, and λ_3 are not necessarily distinct, how does the above condition change?
   (d) Generalize the results that you obtained for the above two cases to the pair (A, B) where

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}.$$

4. Consider the following matrices

$$\begin{aligned} A_1 &= \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}. \end{aligned}$$

   (a) What are the conditions on γ_1, γ_2 and γ_3 for the pair (A_1, B_1) to be reachable?
   (b) Using the matrices A_1, A_2, B_1 and B_2 we can construct the following pair of matrices

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

      i. Suppose λ_1 and λ_2 are distinct, what are the conditions on γ_1, γ_2, γ_3, β_1, β_2, and β_3 for the pair (A, B) to be reachable?
      ii. Suppose $\lambda_1 = \lambda_2$, what are the conditions on γ_1, γ_2, γ_3, β_1, β_2, and β_3 for the pair (A, B) to be reachable?