

Due Tu 09/17/24 (10am; on DEN Webpage)

1. Problem 2.2 from the Hespanha's book (page 20; attached).
2. Consider the unforced mass-spring system

$$m \ddot{y} + g(y) = 0$$

with three different models for the spring force

- **hardening spring:** $g(y) = k(1 + y^2)y$;
- **softening spring:** $g(y) = k(1 - y^2)y$;
- **linear spring:** $g(y) = ky$,

and $k > 0$.

- (a) Determine a state-space representation of this system.
- (b) Find equilibrium points of the above systems. Discuss your observations for three different spring force models.
- (c) Is this system
 - causal,
 - time-varying,
 - linear,
 - memoryless,
 - finite-dimensional?

Explain.

- (d) For three different spring force models with $m = k = 1$, use Matlab to simulate systems' responses from four different initial conditions. Choose these four initial conditions in order to demonstrate different behavior of the models. Plot corresponding results in the phase plane (horizontal axis determined by position $y(t)$, vertical axis determined by velocity $\dot{y}(t)$) and discuss your observations. You should plot four different figures and each of these should contain three phase portraits.
3. Consider the scalar differential equation

$$\dot{x} = x(x^2 - 1).$$

- (a) Determine the equilibrium points of the above system.
- (b) Determine analytical solutions of the linearized systems that are obtained by linearizing the above system around equilibrium points that you found in (a).
- (c) Use Matlab to simulate systems' responses from different initial conditions and provide plots that illustrate how x changes with time. Use your results obtained in (b) to illustrate responses of linearized systems and compare them to the responses of nonlinear system from the same initial conditions.
- (d) The above system can be interpreted as the gradient descent dynamics that can be used to compute the solution of the following unconstrained optimization problem

$$\underset{x}{\text{minimize}} \quad g(x)$$

where $g(x) = x^2/2 - x^4/4$. Indeed, it is easy to verify that the above system can be written as

$$\dot{x} = -\frac{dg(x)}{dx}$$

Explain your computational observations in (c) using this insight.

4. Consider the unconstrained optimization problem

$$\underset{x}{\text{minimize}} \quad g(x) \tag{1}$$

where $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by

$$g(x) := \frac{1}{4} \|xx^T - \Lambda\|_F^2$$

and $\Lambda = \Lambda^T$ is a symmetric matrix. For any matrix M , the Frobenius norm $\|M\|_F$ is given by

$$\|M\|_F^2 = \text{trace}(M^T M) = \text{trace}(M^T M) = \sum_{i,j} M_{ij}^2.$$

If the matrix Λ is positive definite, i.e., if all of its eigenvalues are positive, then the optimal solution x^* to problem (1) corresponds to the best rank-1 approximation to Λ which is given by

$$\hat{\Lambda} := x^*(x^*)^T \approx \Lambda. \tag{2}$$

Suppose that the matrix Λ is given by

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_1 > \lambda_2 > 0$.

- Form the gradient-flow dynamics associated with problem (1), i.e.,

$$\dot{x} = -\nabla g(x). \tag{3}$$

- Find all the equilibrium points of system (3).
- Linearize (3) around each of its equilibrium points.

Hint: There are five equilibrium points.

Does such an equilibrium point always exist?

- (d) Assume that $b = 1/2$ and $mg\ell = 1/4$. Compute the torque $T(t)$ needed for the pendulum to fall from $\theta(0) = 0$ with constant velocity $\dot{\theta}(t) = 1, \forall t \geq 0$. Linearize the system around this trajectory. \square

2.2 (Local linearization around a trajectory). A single-wheel cart (unicycle) moving on the plane with linear velocity v and angular velocity ω can be modeled by the nonlinear system

$$\dot{p}_x = v \cos \theta, \quad \dot{p}_y = v \sin \theta, \quad \dot{\theta} = \omega, \quad (2.11)$$

where (p_x, p_y) denote the Cartesian coordinates of the wheel and θ its orientation. Regard this as a system with input $u := [v \ \omega]' \in \mathbb{R}^2$.

- (a) Construct a state-space model for this system with state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} p_x \cos \theta + (p_y - 1) \sin \theta \\ -p_x \sin \theta + (p_y - 1) \cos \theta \\ \theta \end{bmatrix}$$

and output $y := [x_1 \ x_2]' \in \mathbb{R}^2$.

- (b) Compute a local linearization for this system around the equilibrium point $x^{\text{eq}} = 0, u^{\text{eq}} = 0$.
- (c) Show that $\omega(t) = v(t) = 1, p_x(t) = \sin t, p_y(t) = 1 - \cos t, \theta(t) = t, \forall t \geq 0$ is a solution to the system.
- (d) Show that a local linearization of the system around this trajectory results in an LTI system. \square

2.3 (Feedback linearization controller). Consider the inverted pendulum in Figure 2.6.

- (a) Assume that you can directly control the system in torque, i.e., that the control input is $u = T$.

Design a feedback linearization controller to drive the pendulum to the upright position. Use the following values for the parameters: $\ell = 1 \text{ m}$, $m = 1 \text{ kg}$, $b = 0.1 \text{ N m}^{-1} \text{ s}^{-1}$, and $g = 9.8 \text{ m s}^{-2}$. Verify the performance of your system in the presence of measurement noise using Simulink[®].

- (b) Assume now that the pendulum is mounted on a cart and that you can control the cart's jerk, which is the derivative of its acceleration a . In this case,

$$T = -m \ell a \cos \theta, \quad \dot{a} = u.$$

Design a feedback linearization controller for the new system.

What happens around $\theta = \pm\pi/2$?

Note that, unfortunately, the pendulum needs to pass by one of these points for a swing-up, i.e., the motion from $\theta = \pi$ (pendulum down) to $\theta = 0$ (pendulum upright). \square

Attention! Writing the system in the carefully chosen coordinates x_1, x_2, x_3 is crucial to getting an LTI linearization. If one tried to linearize this system in the original coordinates p_x, p_y, θ with dynamics given by (2.11), one would get an LTV system.