Due Th 10/03/24 (10am; on DEN Webpage)

1. (a) Suppose that A and B are constant square matrices. Show that the state transition matrix for the time-varying system described by

$$\dot{x}(t) = e^{-At} B e^{At} x(t)$$

is

$$\Phi(t,s) = e^{-At} e^{(A+B)(t-s)} e^{As} \; .$$

(b) If A is an $n \times n$ matrix of full rank, show using the definition of the matrix exponential that

$$\int_0^t e^{A\sigma} d\sigma = [e^{At} - I]A^{-1}.$$

Using this result, obtain the solution to the linear time-invariant equation

$$\dot{x} = Ax + B\bar{u} \; , \quad x(0) = x_0$$

where $\bar{u}$ is a constant r-dimensional vector and B is an $(n \times r)$-dimensional matrix.

2. Consider the discrete-time system

$$\begin{aligned} x_{k+1} &= A\, x_k \; + \; B\, u_k \\ y_k &= C\, x_k \; + \; D\, u_k \\ x(k_0) &= x_0 \end{aligned}$$

with constant matrices A, B, C, and D.

(a) Prove that this system is linear and time-invariant.

(b) Using the definition of the $\mathcal{Z}$-transform prove that $\mathcal{Z}\left(A^k\right) = z\,R(z)$, where $R(z) := (z\,I \, - \, A)^{-1}$ is the resolvent of the matrix A.

(c) For

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0.3 \end{bmatrix}$$

determine $R(z)$. From the resulting expression for the resolvent compute the state transition matrix of the above system at $k = 9$.

3. Use the (matrix) exponential series to evaluate e^{At} for:

(a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;

(b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Also, determine the eigenvalue decomposition (i.e., the eigenvalues and eigenvectors) of these two matrices.

Note: You are not allowed to use Matlab in this exercise.

4. Suppose $A(t)$ is an $n \times n$ time-varying matrix with continuous entries that satisfies

$$A(t) \left(\int_{t_0}^t A(\sigma)\, \mathrm{d}\sigma \right) \; = \; \left(\int_{t_0}^t A(\sigma)\, \mathrm{d}\sigma \right) A(t).$$

Show that the state-transition matrix $\Phi(t_1, t_0)$ can be computed as

$$\Phi(t, t_0) \; = \; \exp\left(\int_{t_0}^t A(\sigma)\, \mathrm{d}\sigma \right).$$

5. For the Mathieu equation,

$$\ddot{y}(t) \;+\; (\omega \;-\; \alpha \cos{(2t)})\, y(t) \;=\; 0$$

use Matlab to compute the state-transition matrix with $x_1 = y$, $x_2 = \dot{y}$, $\omega = 2$, $\alpha = 1$. You should do your computations on the time interval of length equal to three periods of oscillations for $t_0 = 0$ and $t_0 = 1$.

6. Consider the following system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \;&=\; \begin{bmatrix} A_{11} & 0 \\ \alpha A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ y \;&=\; \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{S}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are the states, $y \in \mathbb{R}^{n_2}$ is the output, and α is the positive parameter.

(a) Determine if the matrix A is normal.

(b) Find the expression for the state-transition matrix and the resolvent of the above system. The resulting state transition matrix should be partitioned conformably with the partition of the matrix A and its components should be expressed in terms of A_{11}, A_{21}, A_{22}, and α.

(c) For

$$A_{11} \;=\; \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}, \quad A_{22} \;=\; -1, \quad A_{21} \;=\; \begin{bmatrix} 1 & 2 \end{bmatrix}$$

find the right and the the left eigenvectors of the matrix A. You result should be expressed in terms of the parameter α and the left eigenvectors should be normalized to satisfy bi-orthogonality condition.

(d) For the system given in part (c), determine the expression for the system's output arising from the initial conditions in x_1 and x_2. You should use the results obtained in part (c) here.

(e) Sketch the output components determined in part (d). How do your results change if α is increased? Explain your observations.