Due Tuesday 10/29/24 (10am; on DEN Webpage)

1. Consider the system parameterized by positive scalars k and R

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -(1+k^2)/R & k \\ 0 & -(2+k^2)/R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = x_1.$$

   (a) For what values of parameters k and R is this system stable?

   (b) Derive the formula for the eigenvectors of the matrix A.

   (c) Find the solution of the unforced system, i.e., determine the state transition matrix of the matrix A, and discuss influence of parameters R and k on $x_1(t)$ and $x_2(t)$.

   (d) Identify the value of the parameter k and time t at which the largest value of the first state component $x_1(t)$ takes place.

   (e) Plot $x_1(t)$ for $k = 1$ and two different values of R, $R = 1$ and $R = 100$. What can you conclude?

   (f) How would the state equation of the unforced system change if you instead expressed it using the "compressed" time scale, $\tau = t/R$? Discuss your observations.

2. For the LTI system in Problem 1 obtain a suitable Lyapunov function by solving the algebraic Lyapunov equation

$$A^T P + PA = -Q$$

   with

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   What can you conclude about stability properties of this systems based on this exercise?

3. For the LTI system

$$\dot{x} = Ax$$

   with $x(t) \in \mathbb{R}^n$ suppose that there are symmetric positive definite matrices P and Q and a positive scalar μ such that the matricial equation

$$A^T P + PA + 2\mu P + Q = 0$$

   holds. Prove that all eigenvalues of the matrix A have real parts that are smaller than $-\mu$.

   Hint: Start by showing that all eigenvalues of the matrix A have real parts smaller than $-\mu$ if and only if all eigenvalues of the matrix $A + \mu I$ have real parts that are smaller than zero.

4. Let the transfer function of a second-order single-input single-output system be given by

$$\frac{Y(s)}{U(s)} = H(s) = \frac{1}{d(s)}. \tag{1}$$

   A friend of yours identified that $H(s)$ has a pair of complex-conjugate poles $\sigma \pm j\omega$.

   (a) Determine second-order differential equation that governs the evolution of the output y.

   (b) Find a coordinate transformation that brings the matrix A of the state-space model with $x_1 = y$ and $x_2 = \dot{y}$ into the following form:

$$\bar{A} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}.$$

   You are not allowed to use the eigenvalue decomposition of the original matrix A in this exercise.