

Due Thursday 11/21/24 (10am; on DEN Webpage)

1. The Lienard's equation

$$\ddot{x}(t) + f(x(t)) \dot{x}(t) + g(x(t)) = 0$$

describes a broad class of systems. Here,

- $f$  is an even function of  $x(t)$ , and  $f(\cdot) \geq 0$
- $g$  is a monotonically increasing function and  $g(0) = 0$ .

For such systems a Lyapunov function can be constructed as

$$V(x) = \int_0^{x_1} g(\tau) d\tau + \frac{1}{2} x_2^2.$$

Based on this, investigate the stability of the origin for the Van der Pol's equation,

$$\ddot{x}(t) + \epsilon(1 - x^2(t)) \dot{x}(t) + x(t) = 0,$$

where  $\epsilon$  is a positive parameter.

2. For the Problem 4 in HW 3, i.e.,

$$\begin{aligned} \dot{x}_1 &= -\frac{x_2}{1+x_1^2} - 2x_1 \\ \dot{x}_2 &= \frac{x_1}{1+x_1^2} \end{aligned}$$

use the Lyapunov function candidate,

$$V(x) = x_1^2 + x_2^2$$

to study the stability properties of the origin.

3. (a) Determine the Lyapunov function whose derivative along the solutions of an LTI system  $\dot{x} = Ax$  with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

is given by

$$\dot{V}(x_1, x_2) = -2(x_1^2 + x_2^2). \quad (1)$$

- (b) Can you find a Lyapunov function whose derivative along the solutions of an LTI system  $\dot{x} = Ax$  with

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$$

is determined by (1)?

4. For an LTI discrete-time system

$$x_{k+1} = Ax_k$$

- (a) Use the quadratic Lyapunov function

$$V(x_k) = x_k^T P x_k$$

with  $P = P^T > 0$  to derive the conditions for stability. In other words, you should obtain an equivalent of the algebraic Lyapunov equation that we derived in class for continuous-time systems.

- (b) Using the fact that, in continuous-time, the solution to the algebraic Lyapunov equation

$$A^T P + P A = -Q$$

is given by

$$P = \int_0^\infty \Phi^T(t) Q \Phi(t) dt$$

where  $\Phi(t) = e^{At}$  denotes the state-transition matrix, postulate how a solution to the algebraic Lyapunov equation in discrete-time should look like. Prove that your "guess" provides the unique solution to the algebraic Lyapunov equation.

- (c) Use a Lyapunov-based analysis to show that the discrete-time LTI system with  $A = a \in \mathbb{R}$  is stable if and only if  $|a| < 1$ .