Due Monday 12/16/24 (10am; on DEN Webpage)

1. Consider the system parameterized by the scalars k and R

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -(1+k^2)/R & 0 \\ k & -(2+k^2)/R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = x_2.$$

   (a) For what values of k and R is this system stable?

   (b) Derive the formula for the H_2 norm of this system as a function of k and R. Using this formula, plot the H_2 norm as a function of k for $R = 1$ and $R = 1000$, and as a function of R for $k = 2$.

   (c) Find the solution of the unforced system (i.e. determine operator $G(t)$ that maps initial conditions to the output $y(t)$, $y(t) = G(t)x_0$).

   (d) Plot the maximal singular value of $G(t)$ as a function of time (on time interval $t \in (0, 1000)$) for two different cases: a) $R = 1000$, $k = 0$; b) $R = 1000$, $k = 2$. How do these two cases compare to each other. Explain the obtained results.

2. Write a MATLAB program to compute the H_∞ norm of a SISO transfer function using the grid (in frequency) method. Test your program on the function

$$H(s) = \frac{1}{s^2 + 10^{-6}s + 1},$$

and compare your answer to the exact solution (computed by hand using the definition of the H_∞ norm).

3. Consider the following pair of matrices

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}.$$

   (a) Suppose λ_1, λ_2, and λ_3 are distinct, what are the conditions on γ_1, γ_2, and γ_3 for the pair (A, B) to be controllable?

   (b) For a discrete-time system with the above given matrices A and B, $\{\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3\}$, and $\gamma_1 = \gamma_2 = \gamma_3 = 1$, plot the energy (as a function of time for $k \geq 3$) of the minimum energy control necessary to reach the final state $x_f = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Comment your results.

   (c) Suppose λ_1, λ_2, and λ_3 are not necessarily distinct, how does the above condition change?

   (d) Generalize the results that you obtained for the above two cases to the pair (A, B) where

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}.$$

4. Consider the following matrices

$$A_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}.$$

   (a) What are the conditions on γ_1, γ_2 and γ_3 for the pair (A_1, B_1) to be reachable?

   (b) Using the matrices A_1, A_2, B_1 and B_2 we can construct the following pair of matrices

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

      i. Suppose λ_1 and λ_2 are distinct, what are the conditions on γ_1, γ_2, γ_3, β_1, β_2, and β_3 for the pair (A, B) to be reachable?

      ii. Suppose $\lambda_1 = \lambda_2$, what are the conditions on γ_1, γ_2, γ_3, β_1, β_2, and β_3 for the pair (A, B) to be reachable?

5. Solve problem 12.2 below.

12.2 (Satellite). The equations of motion of a satellite, linearized around a steady-state solution, are given by $\dot{x} = Ax + Bu$, where x_1 and x_2 denote the perturbations in the radius and the radial velocity, respectively, x_3 and x_4 denote the perturbations in the angle and the angular velocity, and

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 1 \end{bmatrix}, \qquad B := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The input vector consists of a radial thruster u_1 and a tangential thruster u_2.

(a) Show that the system is controllable from u.

(b) Can the system still be controlled if the radial thruster fails? What if the tangential thruster fails?

(c) For the setup in part (a) with $\omega = 1$, design a state-feedback controller $u = -Kx$ to place the eigenvalues of the closed-loop system at $\{-1, -2, -1 \pm 2\mathrm{j}\}$.

(d) Is the system with with the measurements determined by perturbations in the radius and the radial velocity observable?