EE 587 01/09/23

# Nonlinear Control Systems

## Lecture 1

Today: What is this course about?

Topic: Nonlinear Dynamical Systems

Ex: Static equation : $F(x) = 0$

$\hookrightarrow$ no time dependence

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \left(\mathbb{R}^{n\times 1}\right)$$

$x_i \in \mathbb{R}$ : scalar

$F$ : nonlinear function of vector $x$

Special case: $F(x) = Ax - b$

$A \in \mathbb{R}^{m \times n}$: matrix ($m$ rows; $n$ columns)

$b \in \mathbb{R}^{m}$: vector (with $m$ components)

$Ax - b = 0 \iff Ax = b$: Topic of EE 510 (Linear Algebra)

Ex: Optimization

Unconstrained optimization problem

$$\underset{x}{\text{minimize}} \quad g(x)$$

Optimality condition: (necessary)

$\nabla g(x) = 0$; If $F(x) := \nabla g(x) \Rightarrow$

$$F(x) = 0$$

Challenge: In general, it is difficult to find an explicit solution to

$$\nabla g(x) = 0 \quad (F(x) = 0)$$

One approach: use iterative techniques to find a solution (e.g. gradient descent)

$$x^{k+1} = x^k - \alpha^k \nabla g(x^k)$$ : discrete time

step-size

$$\frac{dx}{dt} = -\nabla g(x)$$ : continuous time (gradient flow)

Dynamical systems described by nonlinear recursive (DT) or ODEs (CT) relations

Q: What are transient and asymptotic properties of nonlinear dynamical systems?

Ex: $F(x) = \Phi(Ax + Bu + b) - x$

$F(x) = 0$ : implicit Neural Network

$x$ : Neural state ($x \in \mathbb{R}^n$)

$\Phi$ : nonlinear activation function

$A, B$ : synaptic weights

$u$ : input

$b$ : bias term

Recurrent NN:

$$\frac{dx}{dt} = -x + \Phi(Ax + Bu + b)$$

In both examples: ODEs of the form

$$\frac{dx}{dt} = f(x) \quad (*)$$

$t$: time ($t \in \mathbb{R}_+$)

$x \in \mathbb{R}^n$

$f: \mathbb{R}^n \to \mathbb{R}^n$ (nonlinear function of $x$)

In discrete time

$$x^{t+1} = h(x^t) \;;\; t = 0, 1, 2, 3, \ldots$$

IF

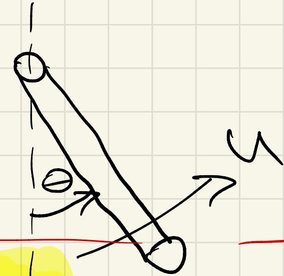
$$\frac{x^{t+1} - x^t}{\Delta t} \approx f(x^t) \Rightarrow$$

Forward Euler discretization (explicit)

$$h(x^t) = x^t - \Delta t \cdot f(x^t)$$

We'll mostly focus on ODEs in EE587 (CT problems)

**Ex:**




2nd Newton's law leads to a nonlinear differential Eq.

$$\boxed{m \cdot l \cdot \underset{\ddot{\theta}}{\frac{d^2\theta}{dt^2}} + l \cdot k \cdot \underset{\dot{\theta}}{\frac{d\theta}{dt}} + m \cdot g \cdot \sin\theta = u}$$

Not in form (*) $\left[ \frac{dx}{dt} = f(x, u) \right]$ external input

Q: How to bring higher order (in this case 2) ONE into the 1st order form (*) (with input u):

$$x_1 = \theta \longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{matrix} \dot{\theta} \\ \ddot{\theta} \end{matrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}\dot{\theta} + \frac{1}{ml} \cdot u \end{bmatrix} \quad (6)$$

$$x_2 = \frac{d\theta}{dt} = \dot{\theta} \longrightarrow$$

Thus: $\dot{x} = f(x, u)$ with $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$f(x, u) = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m} x_2 + \frac{1}{ml} u \end{bmatrix}$$

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Note: For $\underbrace{\text{systems}}_{\text{higher order}}$ [ODEs] without derivatives of input $u$ [wrt time], we can choose "physical state variables" as components of the vector $x$ [phase coordinate form]

$x_1$ = "output" = angle $\theta$

$x_2 = \dfrac{d\,\text{output}}{dt}$ = velocity $\dot{\theta}$

Note: $\dot{x} = f(x,u)$ : $f$ does NOT explicitly depend on time (only function of $x$ and $u$)

time-invariant system

Time-varying systems:

$\dot{x} = f(x, u, t)$ explicit time dependence

Ex: For pendulum:

$m, l, k = \text{const} \Rightarrow$ time-invariant system

otherwise: time-varying

$\hookrightarrow$ If either of them depends on time $t$

In EE 505: (linear systems)

LTI systems: $\dot{x} = Ax + Bu$ $\leftarrow$ A, B constant matrices

Time-varying: $\dot{x} = A(t)x + B(t)u$

$\rightarrow$ solution given by:

$$x(t) = \underbrace{e^{At}x(0)}_{\text{unforced response (response to ICs)}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{forced response (response to inputs)}}$$

Note: For pendulum: principle of superposition does NOT apply!!!

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For nonlinear systems, principle of superposition does NOT hold and very rarely we can write an explicit solution.

☺ We don't want to give up, in spite of this pessimistic statement: we want to study properties of nonlinear dynamical systems by exploiting structural features on nonlinear terms

Ex: Logistic Equation

$$\dot{x} = \alpha \cdot \left(1 - \frac{x}{K}\right) \cdot x$$

growth function (depends on $x$)

$\alpha, K > 0$: positive scalars (constants)

$x(t) \in \mathbb{R}_+$

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Compare with linear model (constant growth function)

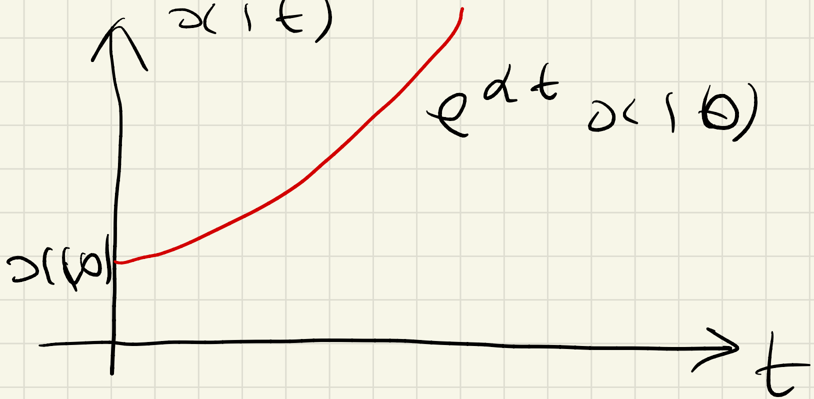
$$\dot{x} = \alpha \cdot x \Rightarrow x(t) = e^{\alpha t} x(0)$$

For $\alpha > 0 \Rightarrow x(t)$ monotonically increasing

$$x(t) \xrightarrow{t \to \infty} \infty$$

(11)

In linear model: only Eq. Point
$\bar{x} = 0$
(unstable)

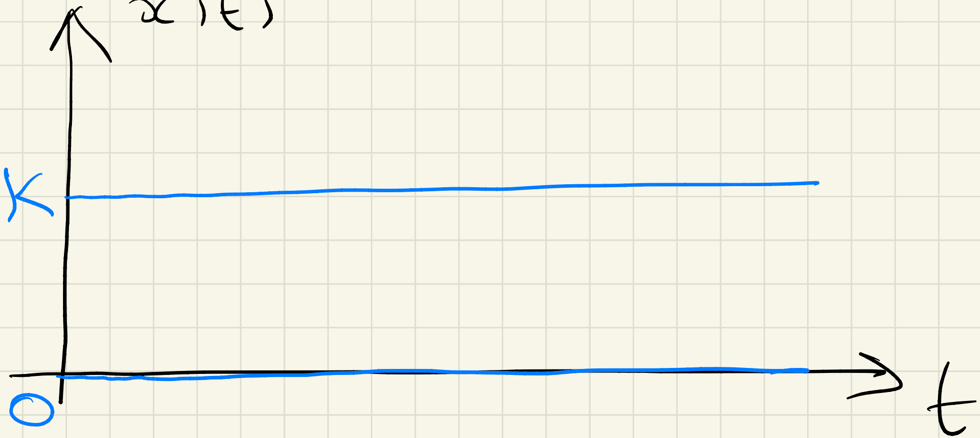



Nonlinear model: Eq. points:

$$0 = \alpha \cdot \left(1 - \frac{\bar{x}}{K}\right) \bar{x} \Rightarrow \boxed{\bar{x}_1 = 0; \ \bar{x}_2 = K}$$

Two Eq. points: zero and K

carrying capacity




Next time: we'll see that nonlinear model saturates the growth of population (reasonable) and we'll "study" features of the response (e.g. stability of Eq. points)