Lecture 2

01/11/23

Last time:

- Class logistics
- State-space Models: $\frac{dx}{dt} = f(x,u)$
- Examples: Pendulum, logistic equation

Today:

- Equilibrium points
- Linearization
- Flows of first order (scalar) systems
  $x(t) \in \mathbb{R}$

Unforced nonlinear System

(no input)

$$\frac{dx}{dt} = f(x) \quad (*) \quad ; \quad x(t) \in \mathbb{R}^n \quad ; \quad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

t : time

$x_i(t) \in \mathbb{R}$

nonlinear function $\quad f : \mathbb{R}^n \to \mathbb{R}^n$

NB Note : extends to matrices

$$\dot{X} = F(X) \quad ; \quad X \in \mathbb{R}^{m \times n}$$

Eq. Points : constant trajectories

i.e. solutions to $(*)$ that do NOT change with time

$x(t) = \bar{x} = \text{const. for all } t$

Eq. points (for $\dot{x} = f(x)$)

$x(t_0) = \bar{x} \Rightarrow x(t) = \bar{x}$ for all t

In particular $\dfrac{d\bar{x}}{dt} \equiv 0 \overset{(*)}{\Longrightarrow}$

$\dfrac{d\bar{x}}{dt} = 0 = f(\bar{x}) \Rightarrow$

We can determine Eq. points by solving nonlinear Equation

$f(\bar{x}) = 0$

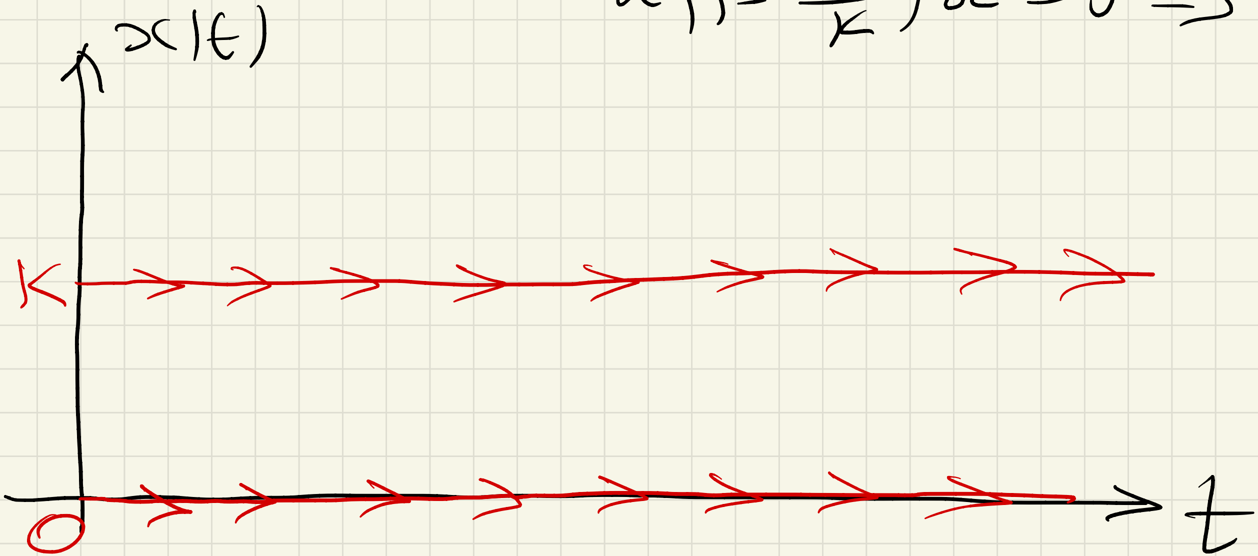
Ex: Logistic Equation

$$\dot{x} = \underbrace{\alpha\left(1 - \frac{x}{K}\right)x}_{f(x)} \quad ; \quad x(t) \in \mathbb{R}_+$$

$\alpha, K > 0$ (constant parameters)

Eq. points: $f(\bar{x}) = 0$

$$\alpha\left(1 - \frac{\bar{x}}{K}\right)\bar{x} = 0 \Rightarrow \begin{array}{l} \bar{x}_1 = 0 \\ \bar{x}_2 = K \end{array}$$



Q: What happens if $x(t_0) \neq \bar{x}_1$ or $\bar{x}_2$

If we had a linear model

$$\dot{x} = \alpha x \Rightarrow \bar{x} = 0$$

$$x(t) = e^{\alpha t} x(t_0) \xRightarrow{\alpha > 0} x(t) \uparrow$$

$$\lim_{t \to \infty} x(t) = \infty$$

In nonlinear model, there is another eq. point that influences dynamical properties of our system (dependence of solutions on time)

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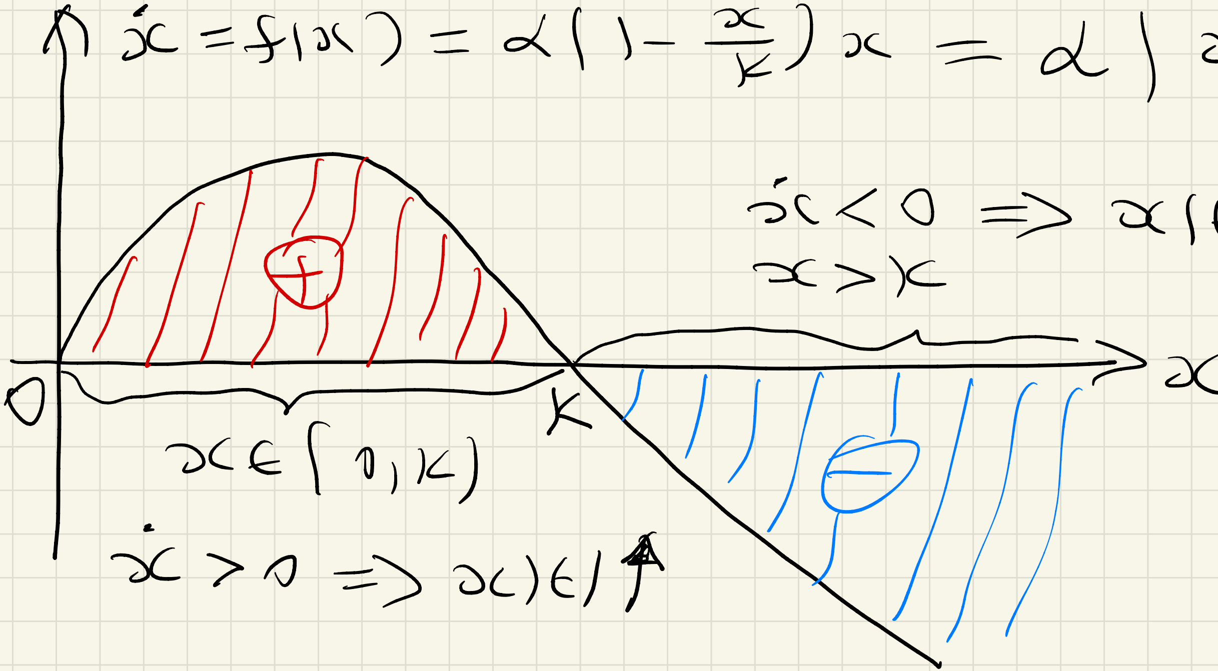
Useful tool (for scalar systems)

Plot of $f(x)$ v.s. x

$$\dot{x} = f(x) = \alpha\left(1 - \frac{x}{K}\right)x = \alpha\left(x - \frac{x^2}{K}\right)$$



$\dot{x} < 0 \Rightarrow x(t) \downarrow$

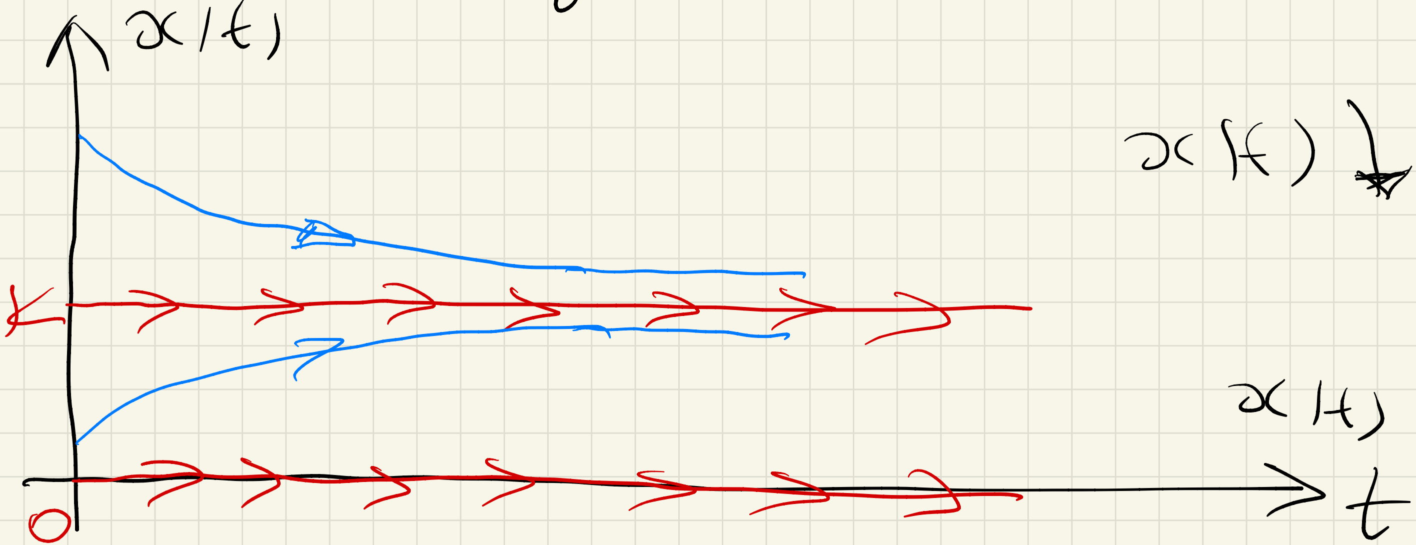
$x > K$

$x \in [0, K)$

$\dot{x} > 0 \Rightarrow x(t) \uparrow$

Based on this:

qualitatively we should have:



Note: We have derived these conclusions without solving differential eq.

We'll formalize this approach in the course

Quantitative analysis necessary to figure out rate of convergence; again, we'll do this without explicitly solving ODEs.
[Lyapunov-based analysis: later in the course

Linearization: tool for local analysis around certain trajectory (does NOT have to be an eq. point). ("life is small")

Given $\dot{x} = f(x)$ and solution $\bar{x}$ [can be time dependent]

decompose $x(t)$ as: $\Longrightarrow$

(8)

$$x(t) = \bar{x}(t) + \tilde{x}(t) \quad \ldots (1)$$

$\bar{x}(t)$: given trajectory (e.g. eq. point)

$\tilde{x}(t)$: perturbation; fluctuation; deviation (around $\bar{x}$)

Substitute (1) to $\circledast$

$$\dot{x}(t) = \boxed{\dot{\bar{x}}(t) + \dot{\tilde{x}}(t) \overset{\circledast}{=} f(\bar{x} + \tilde{x})} \Longrightarrow$$

$$\dot{\tilde{x}}(t) = f(\bar{x} + \tilde{x}) - \underbrace{\dot{\bar{x}}(t)}_{f(\bar{x})} \Longrightarrow$$

Equation for $\tilde{x}(t)$ given by

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Fluctuation Dynamics:

$$\frac{d\tilde{x}}{dt} = f(\bar{x}+\tilde{x}) - f(\bar{x}) \qquad \boxed{*}$$

Eq. for $\tilde{x}$ in which $\bar{x}$ is a coefficient

Note:

- $\boxed{*}$ holds globally; i.e. for $\tilde{x}(t) \in \mathbb{R}^n$ (no approximation made yet)

  and is $\Longleftrightarrow$ to $*$

- If $\bar{x} = \bar{x}(t)$ then $\boxed{*}$ is time-varying even if original system was time-invariant

- If $\bar{x}$ is an eq. point then $\boxed{+}$ is time invariant if $\dot{x} = f(x)$ is time invariant

[Recall : $\dot{x} = f(x)$: T I. $\dot{x} = \alpha \left(1 - \frac{x}{K}\right) x$
$\dot{x} = f(x,t)$: T V $\dot{x} = \alpha(t) \left(1 - \frac{x}{K}\right) x$

If we use Taylor series expansion of f around $\bar{x}$, we have

$O(\|\tilde{x}\|^2)$

$$f(\bar{x} + \tilde{x}) = f(\bar{x}) + \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} \cdot \tilde{x} + H.O.T.$$

$\Rightarrow$

$$f(\bar{x} + \tilde{x}) - f(\bar{x}) \approx \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} \cdot \tilde{x}$$

Thus, linearization around $\bar{x}$ yields:

$$\boxed{\dot{\tilde{x}} = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} \cdot \tilde{x}}$$

Obtained by neglecting H.O.T. in $\tilde{x}$

Holds for arbitrary solution $\bar{x}(t)$ of $\dot{x} = f(x)$; does NOT have to be an eq. point

Ex: Logistic Eq.

$$f(x) = \alpha\left(1 - \frac{x}{K}\right)x = \alpha\left(x - \frac{x^2}{K}\right)$$

$$\frac{\partial f}{\partial x} = \alpha\left(1 - \frac{2x}{K}\right) \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \alpha\left(1 - \frac{2\bar{x}}{K}\right)$$

Clearly, Jacobian depends on $\bar{x}$

In particular for $\bar{x}_1 = 0$ & $\bar{x}_2 = K$

we have:

$$\boxed{A_1} = \frac{\partial f}{\partial x}\Big|_{x=\bar{x}_1=0} = \alpha \cdot \left(1 - \frac{2 \cdot 0}{K}\right) = \boxed{\alpha > 0}$$

$$\boxed{A_2} = \frac{\partial f}{\partial \bar{x}}\Big|_{x=K} = \alpha \cdot \left(1 - \frac{2 \cdot K}{K}\right) = \boxed{-\alpha < 0}$$

$$\dot{\tilde{x}} = \begin{cases} \alpha \tilde{x} & \text{around } \bar{x}_1 = 0 \\ -\alpha \tilde{x} & \text{around } \bar{x}_2 = K \end{cases}$$

$$x(t) = \begin{cases} e^{\alpha t} \tilde{x}(0) & \text{around } \bar{x}_1 = 0 \\ e^{-\alpha t} \tilde{x}(0) & \text{around } \bar{x}_2 = K \end{cases}$$

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Thus, small perturbations around $\bar{x}_1 = 0$ are going to grow [$\bar{x}_1 = 0$ is unstable]

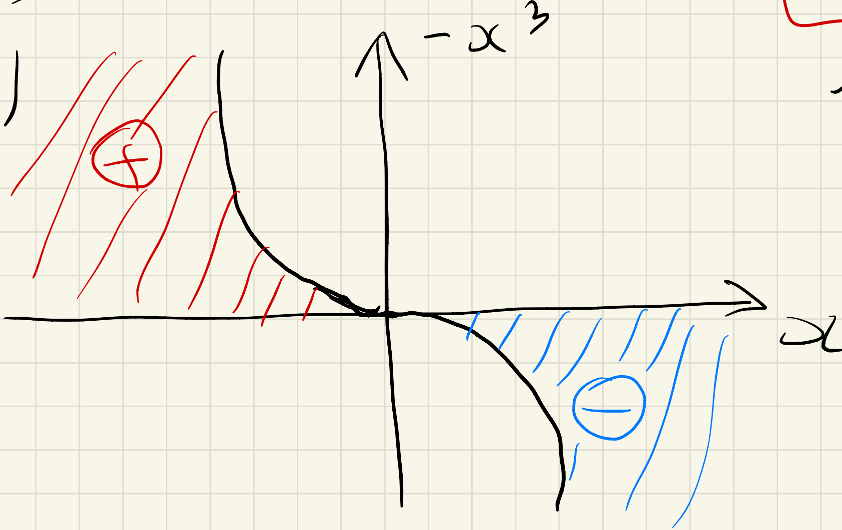
whereas small perturbations around $\bar{x}_2 = K$ are going to decay [$\bar{x}_2 = K$ is locally asymptotically stable]

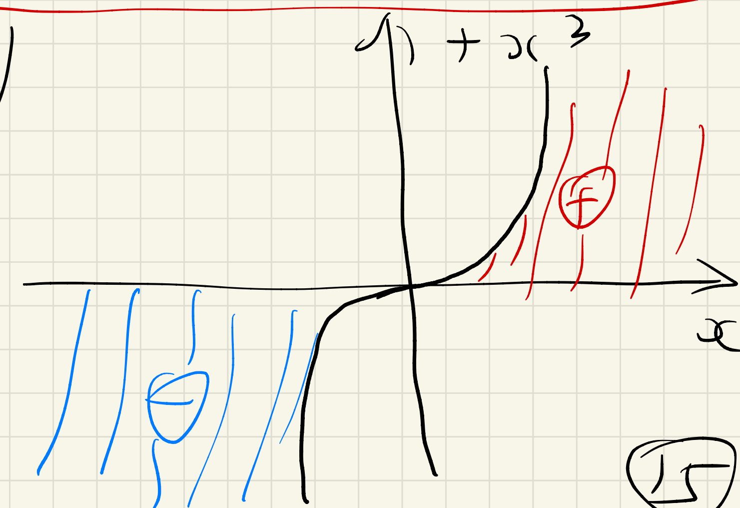
Later, we'll prove that stability of linearization around eq. point $\bar{x}$ guarantees local asymptotic stability of $\bar{x}$ for nonlinear system.

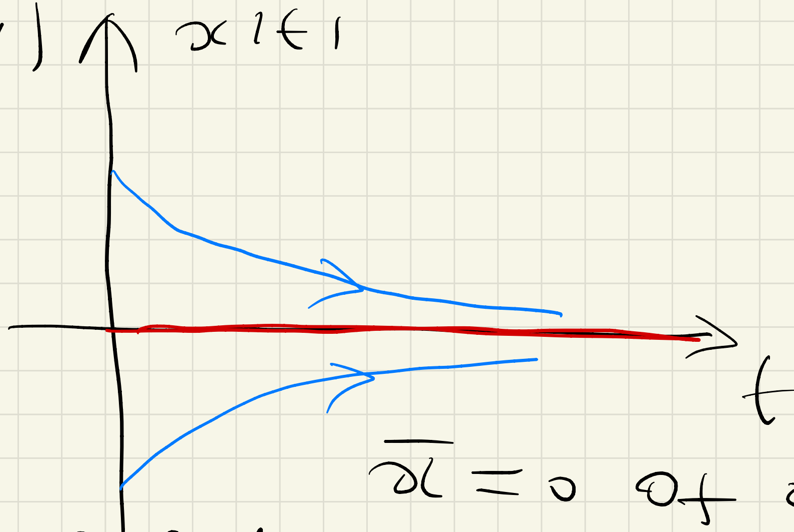
Ex: a) $\dot{x} = -x^3$
b) $\dot{x} = +x^3$ $\Big\} \Rightarrow$ $\bar{x} = 0$ is the unique eq. point

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}} = \mp 3\bar{x}^2 = \boxed{0}$$

$$\Rightarrow \dot{\tilde{x}} = 0 \cdot \tilde{x} \Rightarrow \boxed{\tilde{x}(t) = x(t_0) = \text{const}}$$

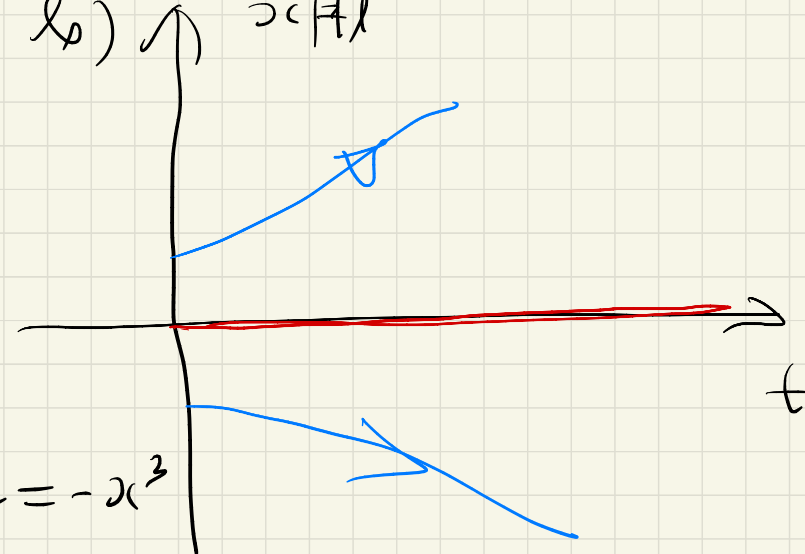
a) 

b) 

a) 

$\bar{x} = 0$ of $\dot{x} = -x^3$

globally asymptotically stable

b) 

UNSTABLE

Note: Linearization CANNOT be used to derive these conclusions

[we'll formalize this]

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Note: For DT systems

$x^+ = f(x)$; $x^+ = x(t+1) = f(x(t))$
$t = 0, 1, 2, \dots$

Eq. points:
$\bar{x} = f(\bar{x})$

[E.g. implicit NN:

$$\bar{x} = \Phi(Ax + b)$$]

Fluctuation Dynamics

$$\bar{x}^+ + \tilde{x}^+ = f(\bar{x} + \tilde{x})$$

$$\tilde{x}^+ = f(\bar{x} + \tilde{x}) - f(\bar{x})$$

Ex: $\dot{x} = \sin(x)$; $x(t) \in \mathbb{R}$

$\sin(\bar{x}) = 0 \Rightarrow \bar{x} = k\pi$; $k = 0, \pm 1, \pm 2, \ldots$

Exercise: Plot $x(t)$ v.s. t

(get insight from visualizing $\sin(x)$ vs. x)

Linearize around $\bar{x} = k\pi$