


Lecture 4

05/23/22

In nonlinear systems we have potentially richer eq. point landscape

unique eq. point: $\dot{x} = -(x - 1)^3$

two isolated eq. points: $\dot{x} = d(1 - \frac{x}{k})x$

infinitely many
(isolated)

$$: \dot{x} = \sin(x)$$

$$: \dot{x} = x^2 + 1 ; x \in \mathbb{R}$$

No real solutions to $\overline{x}^2 + 1 = 0$

$$\frac{dx}{dt} > 0 \text{ for all } x \in \mathbb{R} \Rightarrow \text{D}$$

$\alpha(t)$: monotonically increasing
functions

2° Finite escape time

In nonlinear systems we can have
the following situations

$\|\alpha(t)\| \rightarrow \infty$ in finite time

(i.e. there is $\bar{t} \in \mathbb{R}$ s.t.

$$\lim_{t \rightarrow \bar{t}^-} \|\alpha(t)\| = +\infty$$

Linear system: $\dot{\alpha} = a\alpha \Rightarrow \alpha(t) = e^{at}\alpha_0$
 $a > 0 \Rightarrow |\alpha(t)| \xrightarrow{t \rightarrow \infty} \infty$ ②

$$Ex: \dot{x} = x^2 ; \quad x(t) \in \mathbb{R}$$

Eq. point: $\dot{x} = 0$

$$\int_{x_0}^x \frac{dx}{x^2} = \int_{t_0}^t dt \Rightarrow x(t) = \frac{1}{\frac{1}{x_0} - (t - t_0)}$$

Time irreversibility \Rightarrow can set $t_0 = 0$

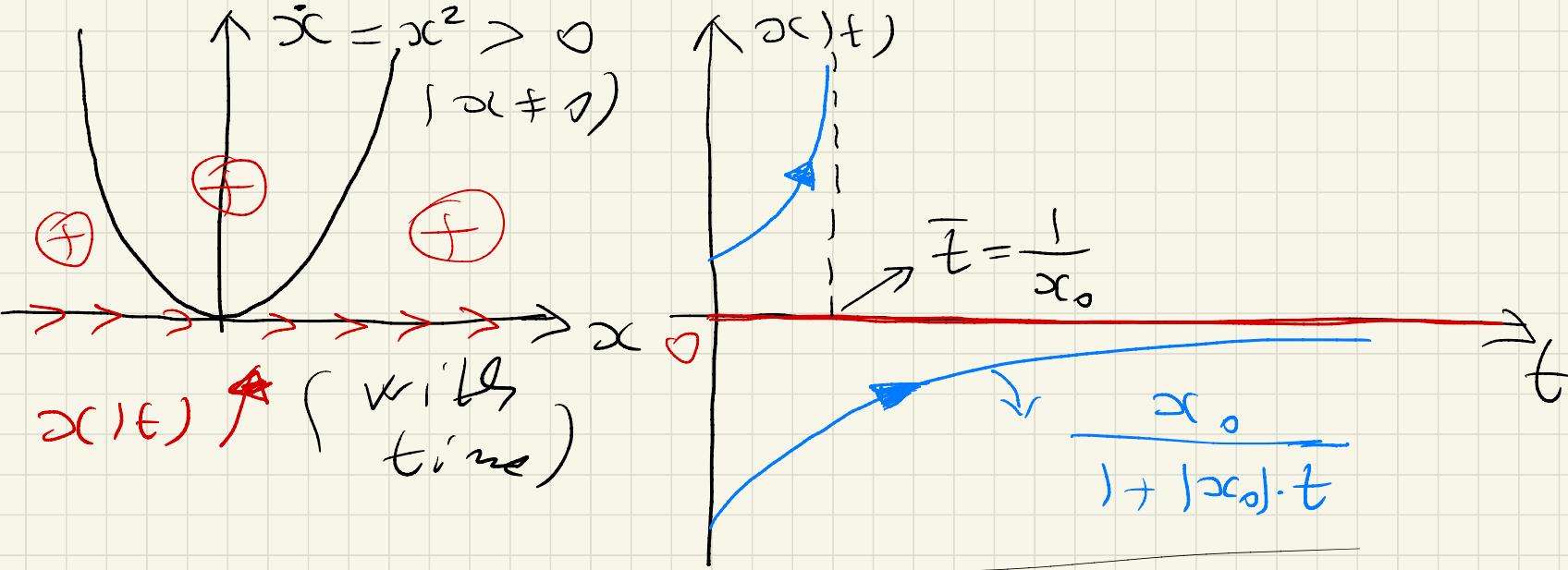
$$x(t) = \frac{x_0}{1 - x_0 t}$$

$$t \geq 0$$

$$\text{For } x_0 > 0 \Rightarrow$$

$$\text{For } x_0 > 0$$

$x(t)$ $\xrightarrow{t \rightarrow \left(\frac{1}{x_0}\right)^-}$ +∞ solution exists
only for $t \in [0, \frac{1}{x_0})$ ③



We could NOT encounter finite escape time in $\dot{x} = Ax$

$$\lim_{t \rightarrow (\frac{1}{x_0})^-} x(t) = \frac{\text{for}}{|x_0| > 0} + \infty$$

If A aside: DT LTS : $x(t+1) = \alpha x(t)$
 $x(t) \in \mathbb{R}^n$

$$x(t) = \alpha^t x(0)$$
$$t = 0, 1, 2, \dots$$

In optimizations: exponential
(in DT)
or geometric (in DT) convergence
rate is called linear convergence
rate.

If convergence rate is
not exponential it's called
sublinear (in optimization) (5)

3° Limit cycles (isolated periodic orbits)

Recall: Oscillation CAN'T happen in scalar (1st order) systems i.e CT
 $\dot{x} = f(x)$; $x(t) \in \mathbb{R}$ (scalar state)

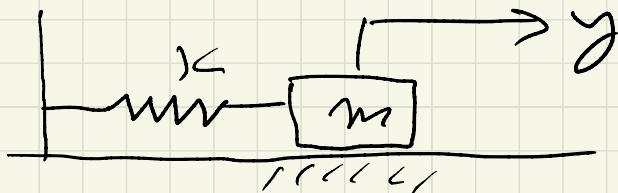
Ex: $\dot{x} = \sin(x)$; $\ddot{x} = x^2$; $\ddot{x} = \pm x^3$
 $\ddot{x} = \alpha(1 - \frac{x}{x_0})x$ ---

Monotonic growth or decay with time (between eq. points)

i.e., CAN'T have

$$x(t) = x(t + \pi) \text{ for } \pi > 0 \quad (6)$$

Ex :



No friction

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{\omega^2}{m} & 0 \end{bmatrix}$$

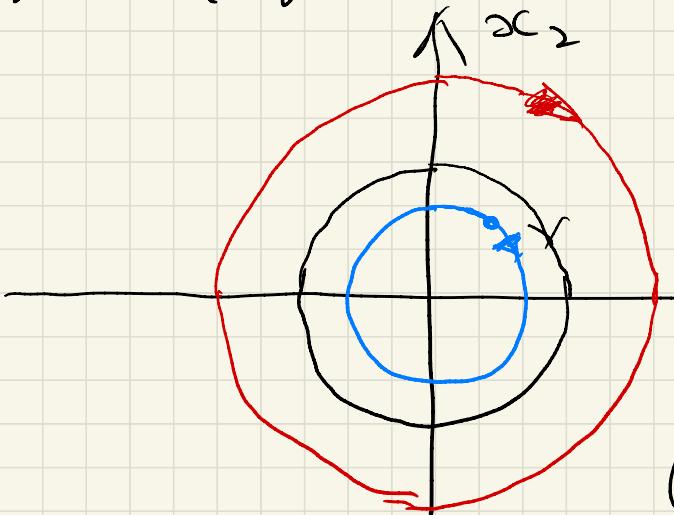
Harmonic oscillator

Please partition

$$\begin{cases} \dot{x}_2 \text{ v.s. } x_1) ; \\ (\omega = \sqrt{m/k}) \end{cases}$$

Key observations:

- Susinately many periodic orbits (No isolated ones)



x_1 , Amplitude of oscillations

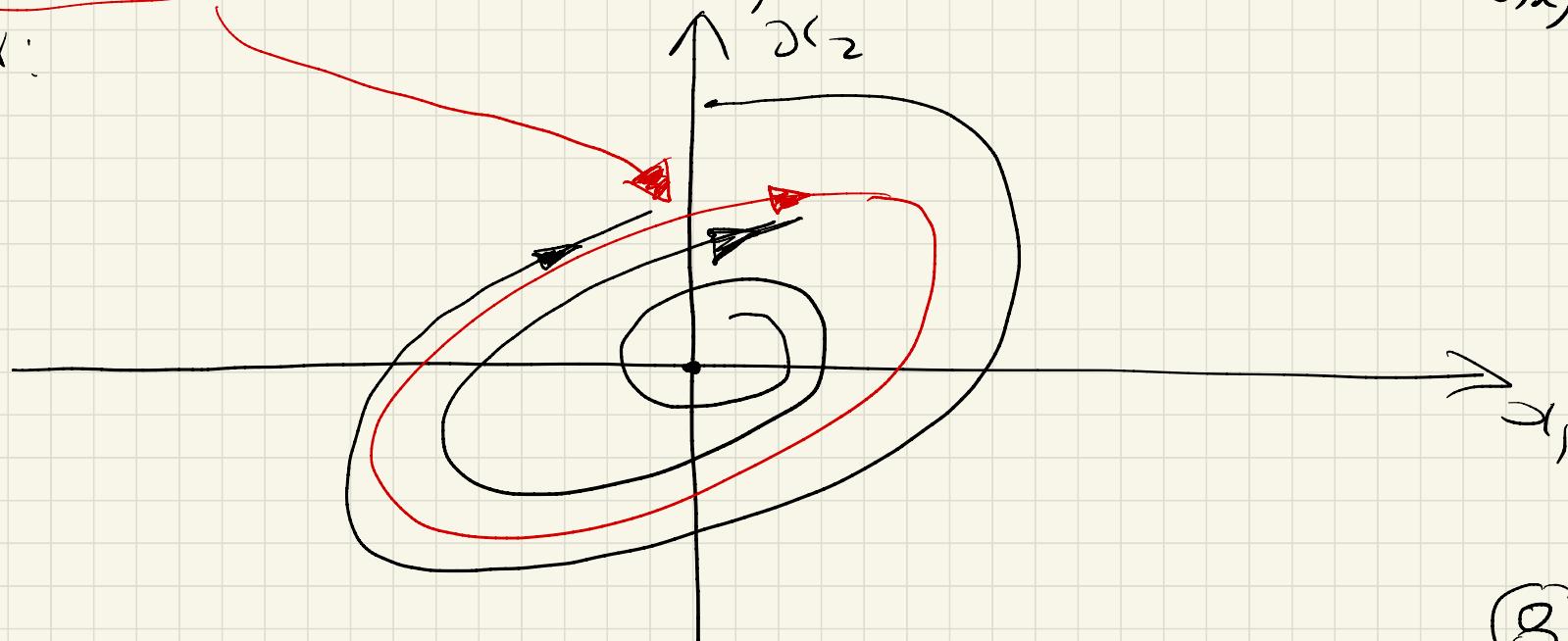
(7)

(depends on initial cond.)

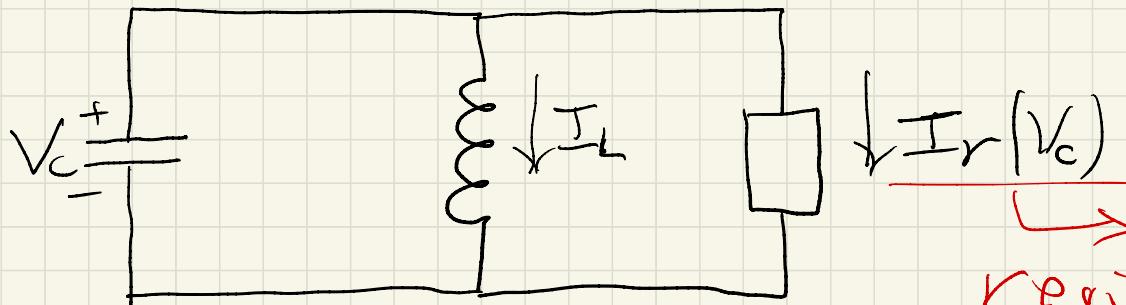
Limit cycles CANNOT happen in
LTI systems; $\dot{x} = Ax$

Round (sustained) isolated periodic
orbits are ONLY possible if $\dot{x} = f(x)$

Ex:



Ex: Van der Pol Oscillator



2nd order

system

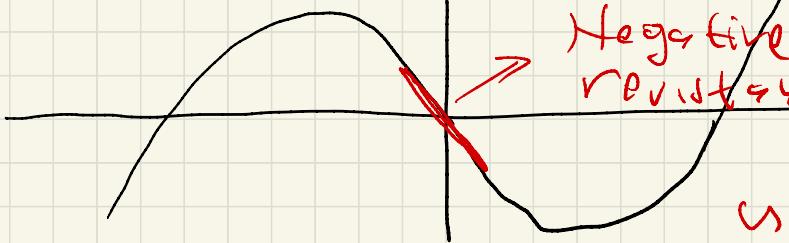
Nonlinear
resistive element

$$\dot{I}_L = \frac{1}{L} \cdot V_c$$

$$\dot{V}_c = -\frac{1}{C} \cdot I_L + \frac{1}{C} (V_c - V_c^3)$$

$- I_{r(V_c)}$

$I_{r(V_c)}$



Negative
resistance

unstable

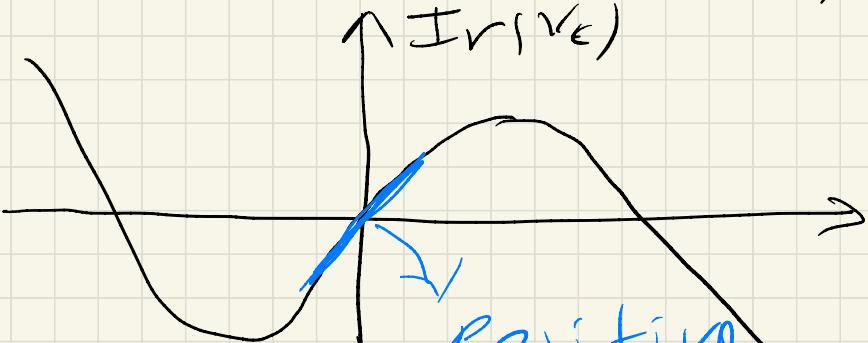
$$\bar{x} = \begin{bmatrix} \bar{I}_L \\ \bar{V}_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

unique eq. exist
Linearization
around $\bar{x} = 0$

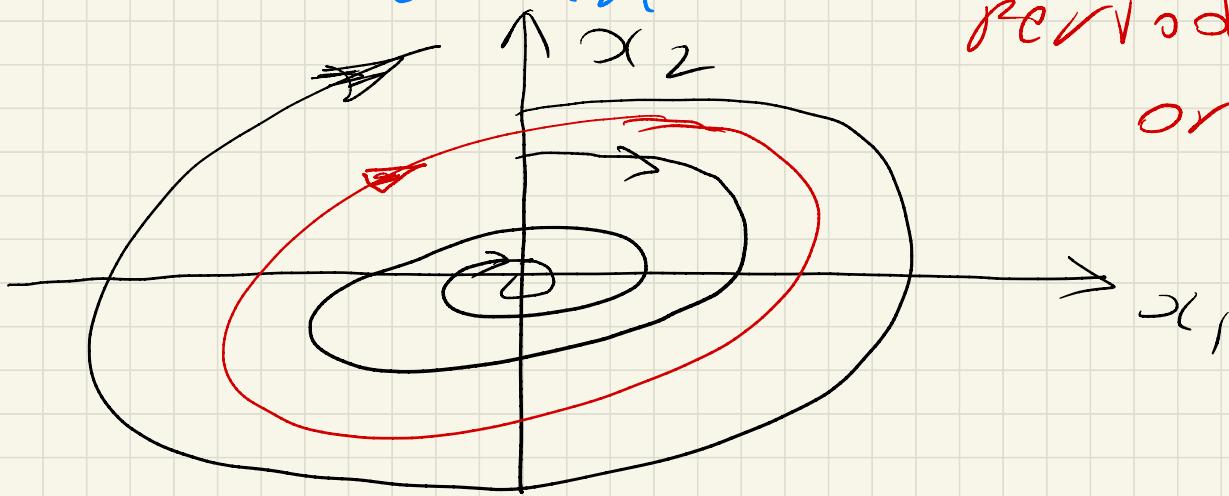
$$A = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & \frac{1}{C} \end{bmatrix}$$

+

If instead we had



positive
resistive
element



$$A = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix}$$

v_c locally asympt.

Stable $\Im s = 0$;

Unstable

periodic

orbit