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# Lecture 5

01/30/23

Last time:

- Multiple isolated E<sub>z</sub>. points
- Finite escape time
- Limit cycles

Today:

- Chaos
- Bifurcations: change of dynamical properties as parameters are varied

4° Chaos: No simple characterization of asymptotic properties of  $\dot{x} = f(x)$ ;  
(as  $t \rightarrow \infty$ )

HIGH sensitivity to initial conditions  
(extreme)

Ex: Lorenz attractor: time-invariant system with 3 states:  $x, y, z$   
coordinates

$$x_1, x_2, x_3$$

3 constant parameters  
 $a, b, c$

$$\dot{x} = a \cdot (x - y)$$

$$\dot{y} = x \cdot (b - z) - y$$

$$\dot{z} = x \cdot y - c \cdot z$$

Exercise:

simulate for:

$$a=10; b=28; c=\frac{8}{3}$$
 ②

Note; zero is a eq. point & of linearization around zero is given by

$$A = \left[ \begin{array}{cc|c} a & -a & 0 \\ b & -1 & 0 \\ 0 & 0 & -c \end{array} \right]$$

$$\det(sI - A) = \det \begin{bmatrix} s-a & a \\ -b & s+1 \end{bmatrix} =$$

$$= s^2 - as + s - a + ab =$$

Local eqm/locally

$$= s^2 + (1-a)s + a(b-1)$$

Stability of zero is given by

$$\begin{cases} a < 1 \\ a(b-1) > 0 \end{cases}$$

Note: For  $a > 1$ ; unstable zero e.g. exist  
and there is no simple way to  
characterize asymptotic properties.

In C.T. systems, we need at least  
3 states for chaos [i.e. CAN'T  
happen in systems (No inputs) with  
2 unscaled states]

scalar states or systems with 2 states]

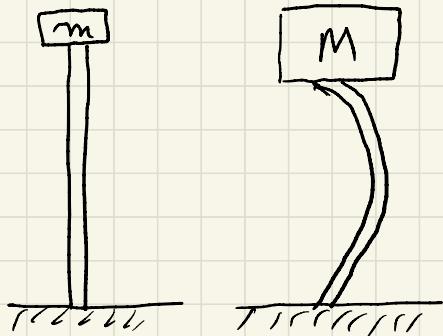
In D.T., chaotic behavior is possible  
even for scalar & Jitterless:

$$x^{k+1} = f(x^k); \quad x^k \in \mathbb{R}$$

(see Myrati's notes for an example) (2)

Bifurcations: appearance or disappearance of eq. points; change of stability properties of eq. points  $\Rightarrow$  parameters change their values

Ex: Buckling Beam



If mass becomes larger than critical value,  
the beam can no longer support the weight  
(i.e., it buckles)

Even in systems with scalar state ( $x(t) \in \mathbb{R}$ ), presence of parameters can lead to "interesting" responses

Three types of bifurcations for scalar systems:

$$1^{\circ} \text{ Fold} : \dot{x} = \lambda \pm x^2$$

$$2^{\circ} \text{ Transcritical} : \dot{x} = \lambda x \mp x^2$$

$$3^{\circ} \text{ Pitchfork} : \dot{x} = \lambda x \mp x^3$$

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$x(t) \in \mathbb{R}$  scalar state

$\lambda \in \mathbb{R}$  : real number (parameter) (6)

Want to study what happens as we change  $\alpha$  (from  $-\infty$  to  $+\infty$ )?

1° Fold:  $\dot{x} = \alpha + x^2$ ;  $x(t) \in \mathbb{R}$

Eg. points:  $\dot{x}^2 + \alpha = 0 \Rightarrow \dot{x}^2 = -\alpha$

For  $\alpha > 0$ : No eg. points

For  $\alpha \leq 0$ :  $\dot{x} = \pm \sqrt{|\alpha|}$  }  $\Rightarrow$

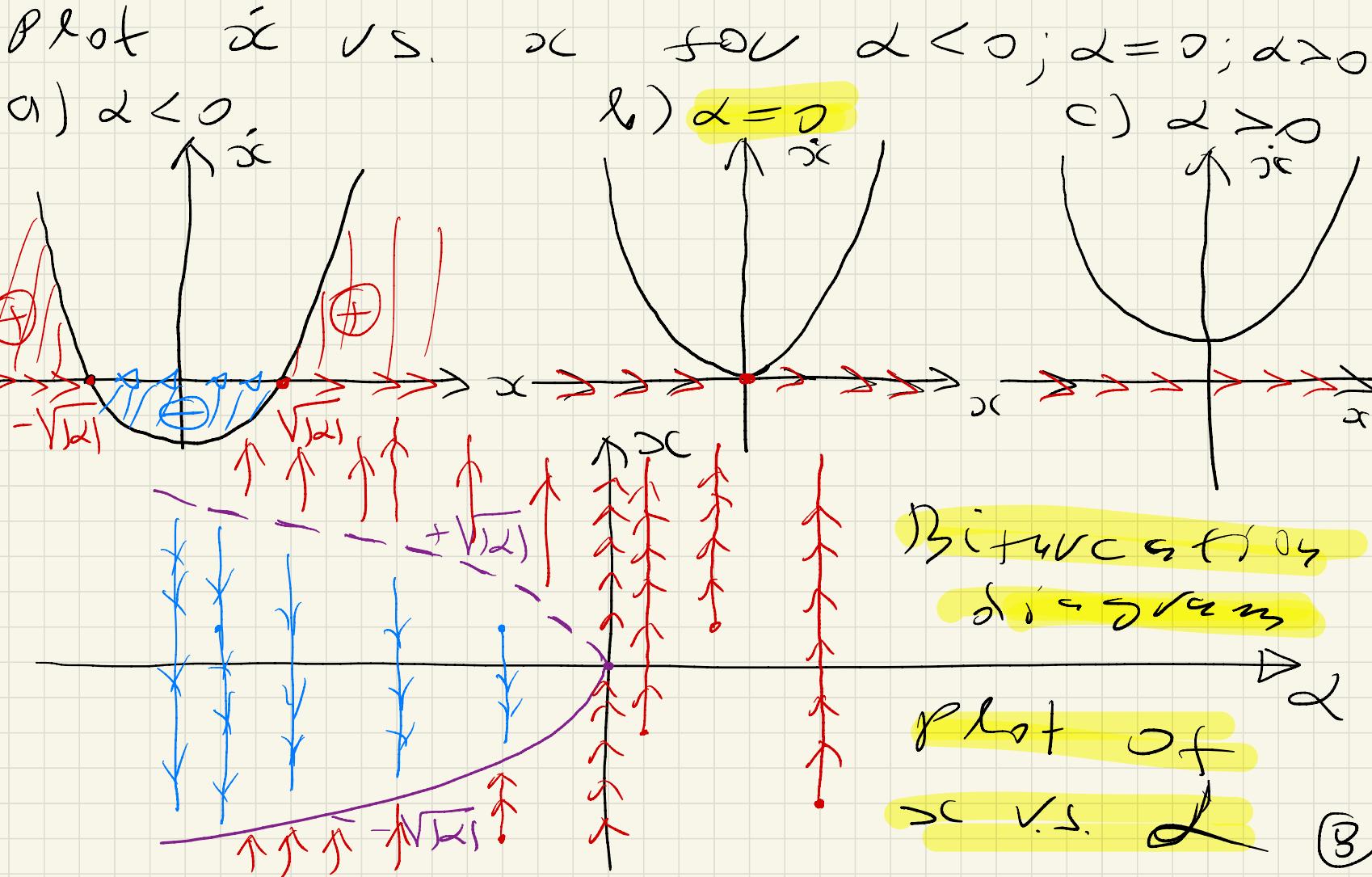
For  $\alpha < 0$ , we have 2 eq. points

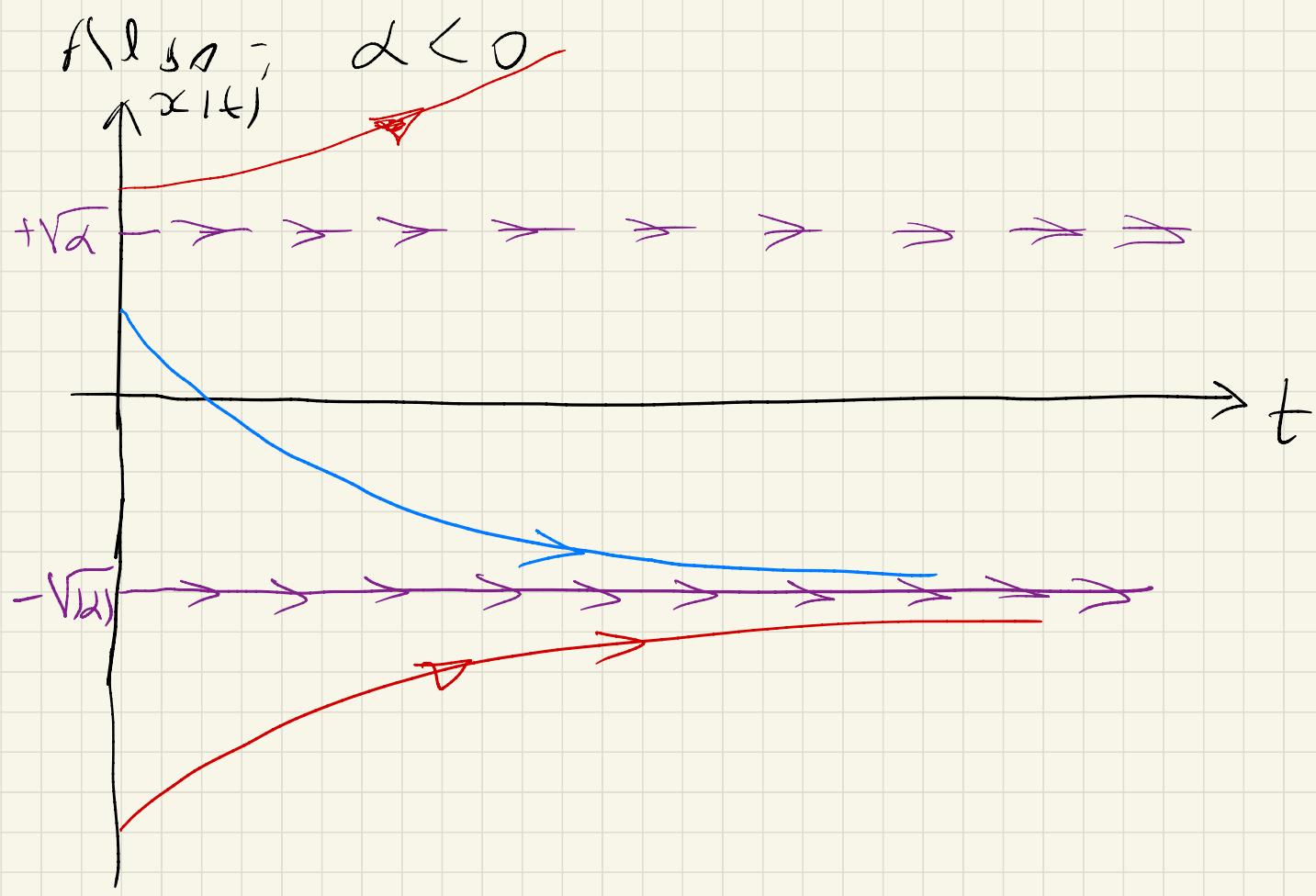
$\alpha = 0$ , — || — ) eq. points

$\alpha > 0$ , — || — ) No eq. points

Note: also called BLUE SKY

⑦





Note:  $\lambda_c = 0$ ; critical value of  $\lambda$   
 [bifurcation point]: two eq. points  
 collapse to a single eq. point  
 and then they dis~~appear~~

similar story for

$$\text{Ex: } \dot{x} = \lambda - x - e^{-\lambda x}$$

$$\text{Eq. points: } \boxed{\lambda - \bar{x} = e^{-\bar{x}}}$$

$$\dot{x} = \lambda - x^2$$

No explicit  
solutions

Use deMoivre's of  $e^{-\lambda x}$  to study  
our problem:

$$\begin{aligned} \dot{x} &= \lambda - x - \sum_{k=0}^{\infty} \frac{(-\lambda x)^k}{k!} = \lambda - x - \left(1 - x + \frac{\lambda x^2}{2} - \dots\right) \\ \dot{x} &= \lambda - 1 - \frac{\lambda x^2}{2} + O(x^3) \approx \boxed{\lambda - 1 - \frac{1}{2}x^2} \end{aligned}$$

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$\Rightarrow$  Around  $x=0$  our system behaves as  $\dot{x} = a - \frac{x^2}{2}$  where  $a = \alpha - 1$

2° Transcritical [Occurs at  $x=0$ ]

$$\dot{x} = \alpha x - x^2 = x(\alpha - x)$$

Eq. points:  $\dot{x}_1 = 0$ ;  $\dot{x}_2 = \alpha$

In contrast to FOLD we have eq. points for any value of  $\alpha$

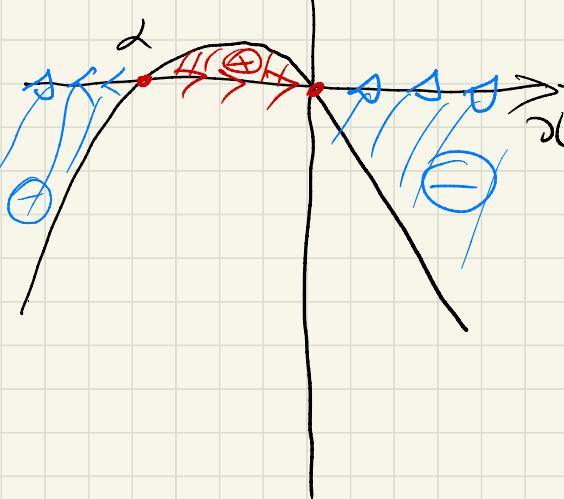
$\alpha \neq 0 \Rightarrow$  2 eq. points

$$\begin{aligned}\dot{x}_1 &= 0 \\ \dot{x}_2 &= \alpha\end{aligned}$$

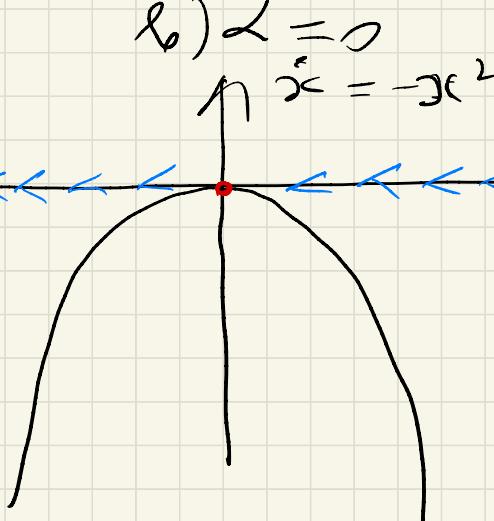
$\alpha = 0 \Rightarrow \dot{x} = 0$  unique eq. point

Next:  $\dot{x}$  vs  $x$  for  $\alpha < 0$ ;  $\alpha = 0$ ;  $\alpha > 0$

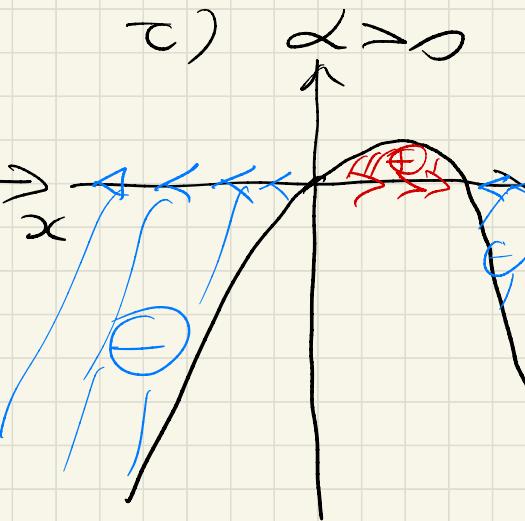
a)  $\alpha < 0$



b)  $\alpha = 0$



c)  $\alpha > 0$



Please convert this info  
to bifurcation diagrams  
(plot of  $\dot{x}$  vs  $\alpha$ )