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## Lecture 6

02/05/23

Last time:

Bifurcations in 1st order systems

- Fold (covered)
- Transcritical (did most of it)
- Pitchfork (today)

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2nd half:

Please portmises of 2nd  
order LTI systems

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## 2° Transcritical

$$\ddot{x} = \lambda x + x^2$$

$$\dot{x} = x(\lambda - x) \Rightarrow$$

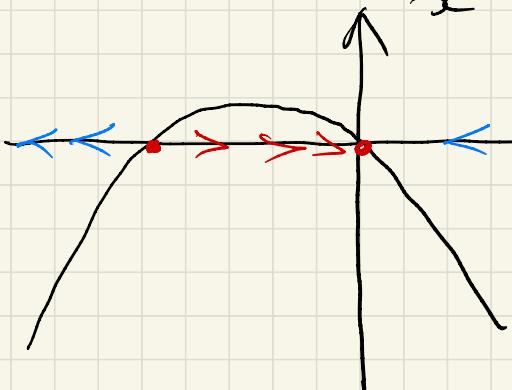
$$\bar{x}_1 = 0$$

$$\bar{x}_2 = \lambda$$

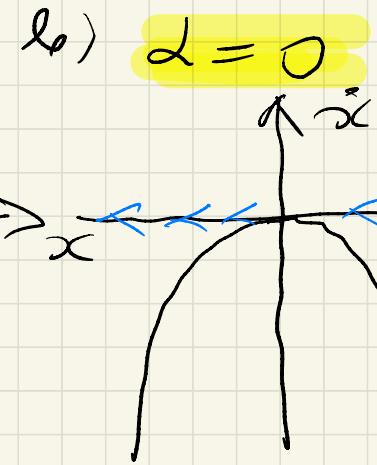
Can do analysis of the signs  
of  $\dot{x}$  v.s.  $x$  and plot bifurcation  
diagram

**critical parameter**

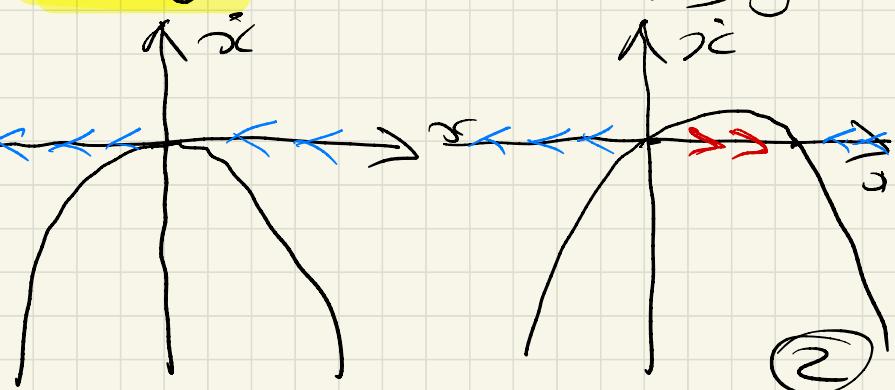
a)  $\lambda < 0$



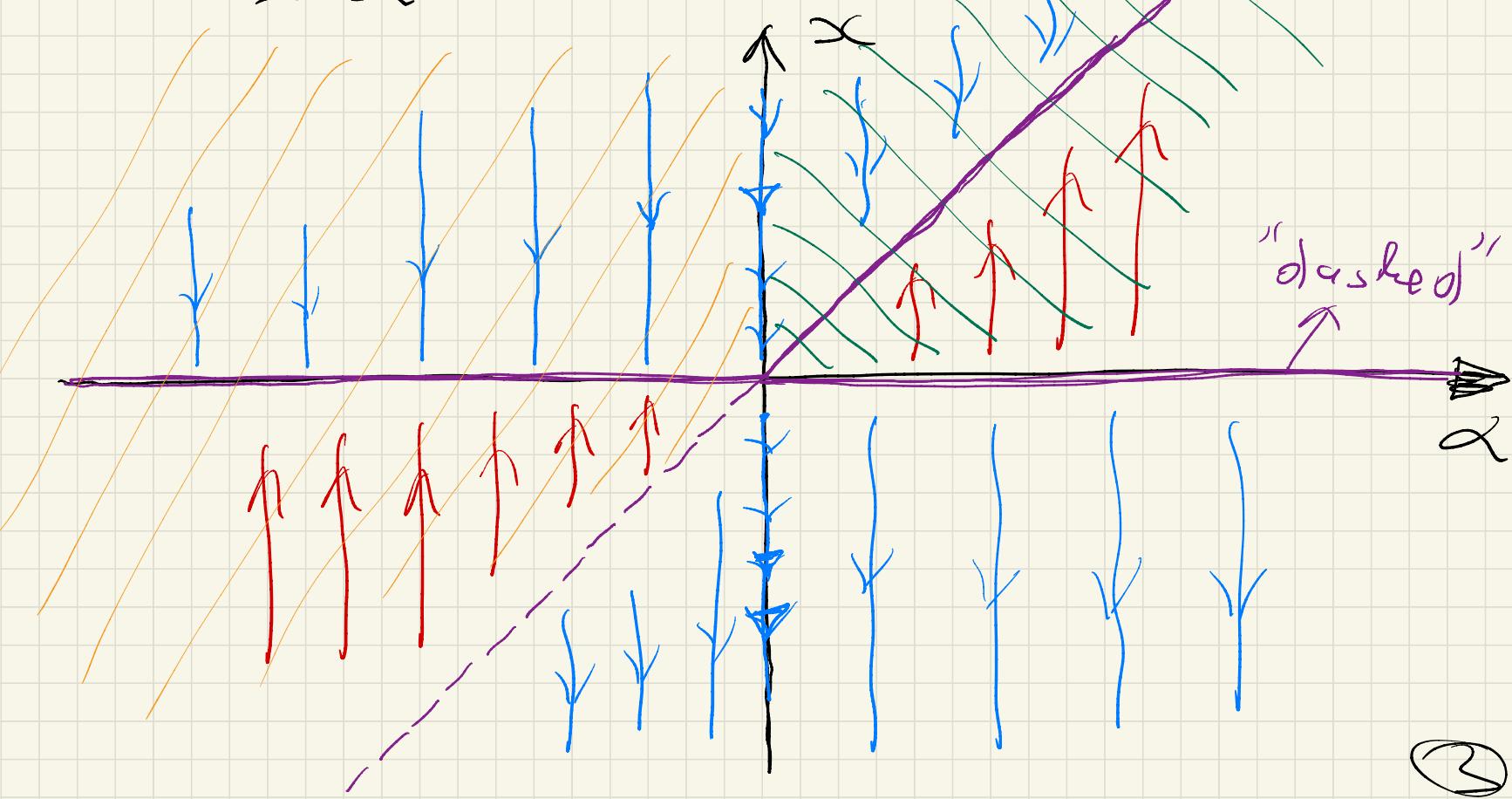
b)  $\lambda = 0$



c)  $\lambda > 0$



Need to translate  $\alpha_1, \alpha_2, c$  to  
 $\alpha$  vs  $L$  plot



Bifurcation diagram:

||||: denotes set of I.C.'s  $x(0)$  for  
 $\lambda < 0$  that converges to  $\bar{x} = 0$   
(region of attraction of a stable  
equilibrium point)

|||: — ) ) — — ) ) —  
 $\lambda > 0$  that converges to  $\bar{x} = \lambda$   
 $(x(0) > 0) \Rightarrow \lim_{t \rightarrow \infty} x(t) = \lambda > 0$

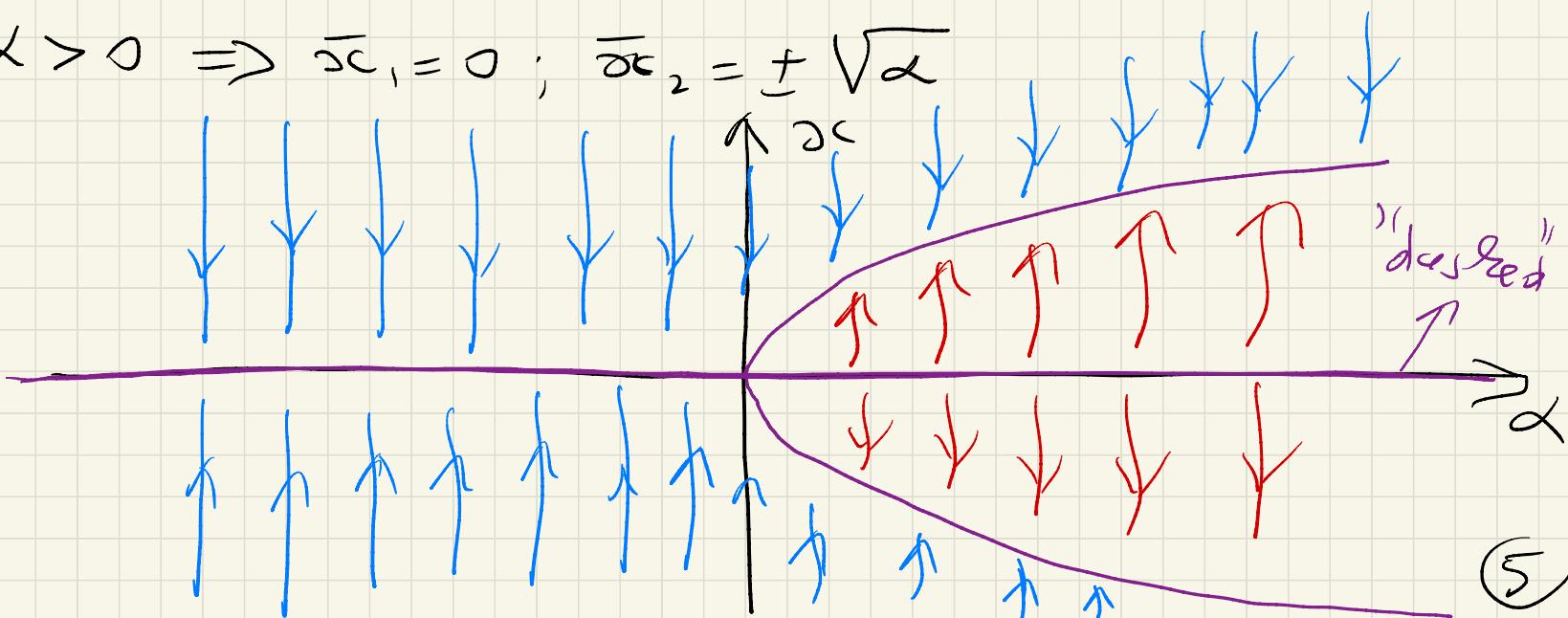
$$3^{\circ} \text{ a) } \dot{x} = \alpha x - x^3 = x(\alpha - x^2) \quad \text{(smiley face)} \quad \text{supercritical}$$

$$\text{b) } \dot{x} = \alpha x + x^3 = x(\alpha + x^2) \quad \text{(frowny face)} \quad \text{subcritical}$$

$$\text{a) } \bar{x} = 0 ; \bar{x}^2 = \alpha \quad \parallel \quad \dot{x} = \alpha x - x^3$$

$\alpha \leq 0 \Rightarrow \bar{x} = 0$  unique eq. point

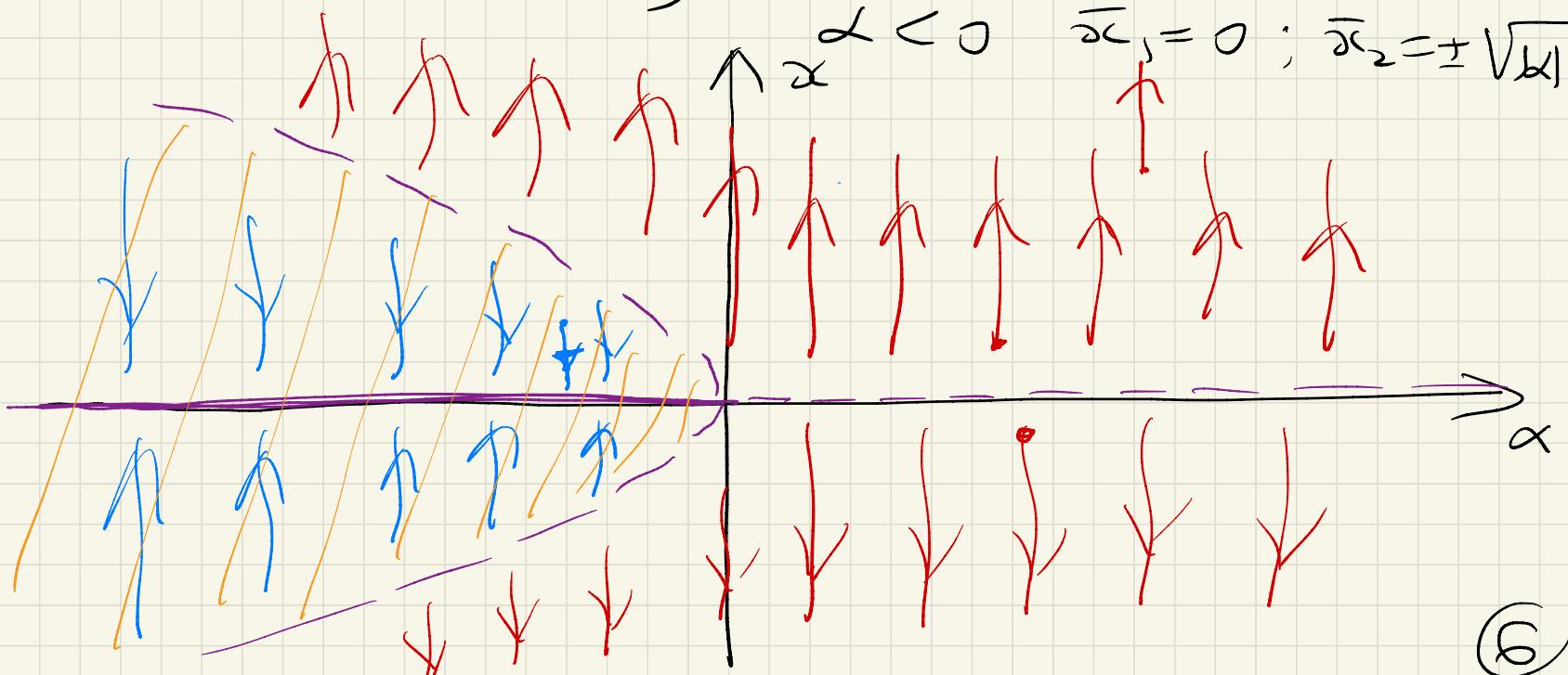
$$\alpha > 0 \Rightarrow \bar{x}_1 = 0 ; \bar{x}_2 = \pm \sqrt{\alpha}$$



(5)

(ii) : Loss of stability of  $\bar{x} = 0$  for  $\lambda > 0$  did NOT result in a blow-up of solutions.

b)  $\dot{x} = x(\lambda + x^2)$ ;  $\lambda \geq 0$   $\bar{x} = 0$  unique



:( Loss of stability of  $\bar{x} = 0$  for  $\lambda \geq 0$  leads to blow up of trajectories for say  $x(0) \neq 0$ .

Next week: Hopf bifurcations  
(2nd order systems)

Phase portraits of 2nd order  
(T) systems

$$\dot{x} = Ax ; x(t) \in \mathbb{R}^2 ; A \in \mathbb{R}^{2 \times 2}$$

$\xrightarrow{\text{const. } t \in \mathbb{R}}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad a_{ij} \in \mathbb{R}$$

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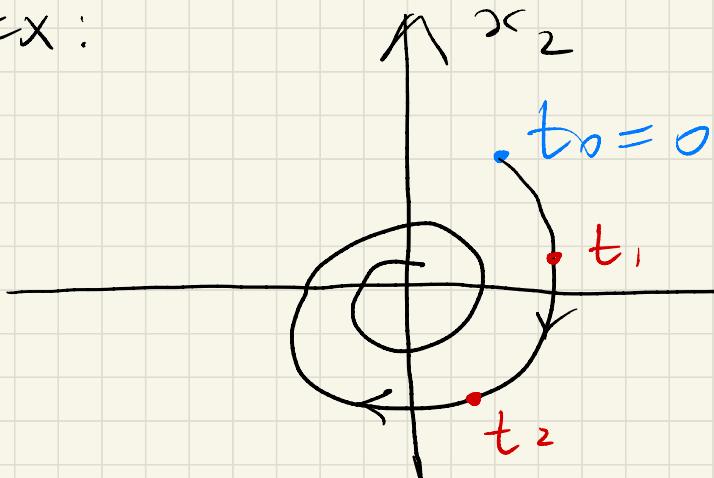
Solutions to  $\dot{x} = Ax$  gives by

$$x(t) = e^{At} x(0)$$

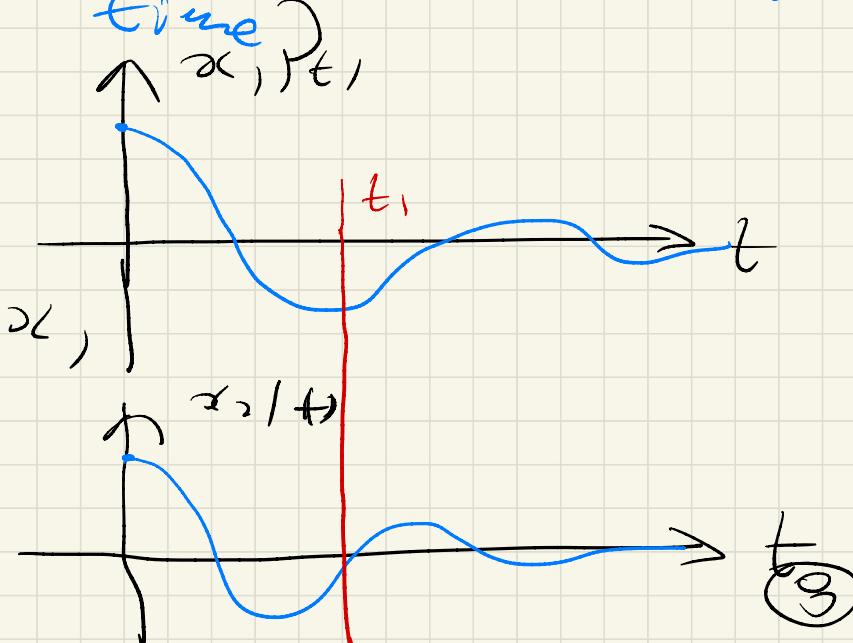
state transition matrix (Matrix Exp.)

Phase portrait: Plot of  $x_2$  vs  $x_1$  (parameterized by time)

Ex:



$$t_0 < t_1 < t_2$$



Change of variables:

$$\dot{x} = Ax$$

(coordinate transformation)

$$x(t) = \Pi z(t); z(t) \in \mathbb{R}^2; \Pi \in \mathbb{R}^{2 \times 2}$$

$\det(\Pi) \neq 0$ : invertible matrix

$$z(t) = \Pi^{-1} x(t)$$

CAN go back & forth between  $x$  &  $z$

$$\underbrace{\Pi \cdot \dot{z}}_{\dot{x}} = A \cdot \underbrace{\Pi \cdot z}_x \Rightarrow \dot{z} = \underbrace{\Pi^{-1} \cdot A \cdot \Pi \cdot z}_{A}$$

$$\dot{z} = \bar{A} \cdot z$$

Three cases of interest:

a)  $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix};$$

$\lambda_1, \lambda_2$ : e-values

of  
(real)

b)  $\lambda, \beta \in \mathbb{R}$

$$\bar{A} = \begin{bmatrix} \lambda & -\beta \\ \beta & \lambda \end{bmatrix};$$

$$\lambda = \alpha \pm j\beta$$

complex

Jordan  
form

c)  $\bar{z}_i = \lambda_i z_i; i=1, 2 \Rightarrow z_i(t) = e^{\lambda_i t} z_i(0)$

Assume:  $\lambda_1 \neq 0$

they are limit of algebraic

solutions:

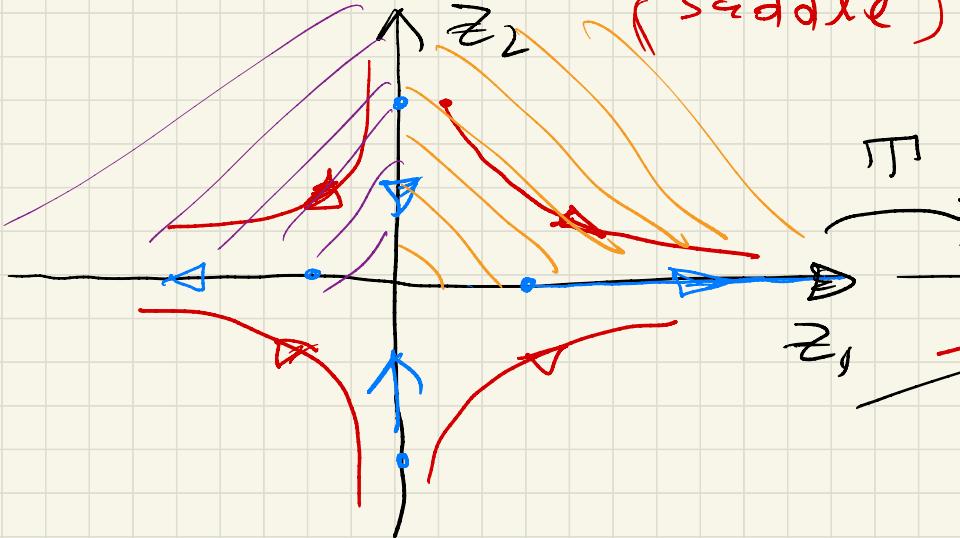
$$z_2 = C \cdot z_1 \frac{\lambda_2}{\lambda_1}$$

No coupling therefore  
 $z_1(t) \Leftrightarrow z_2(t)$

decreasing or

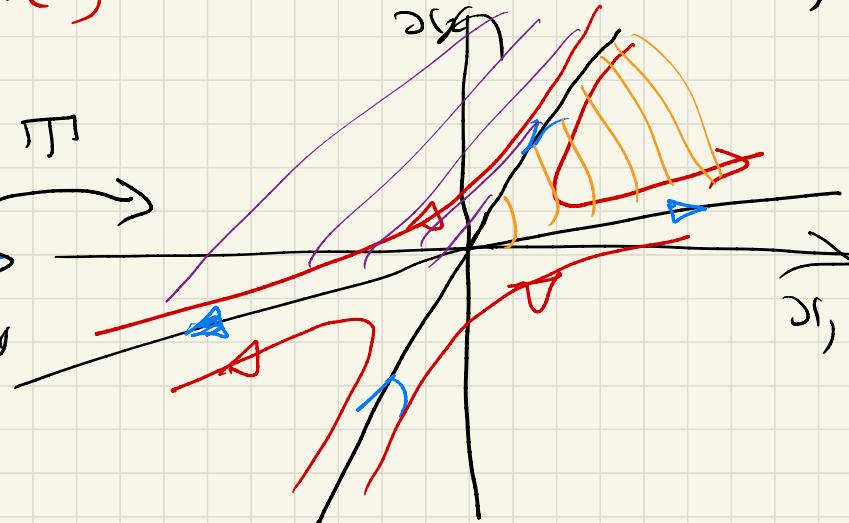
if  $C \in \mathbb{R}$ ,  $z_1(0)$  &  $z_2(0)$

Q1) Let  $\lambda_1 > 0$ ;  $\lambda_2 < 0$



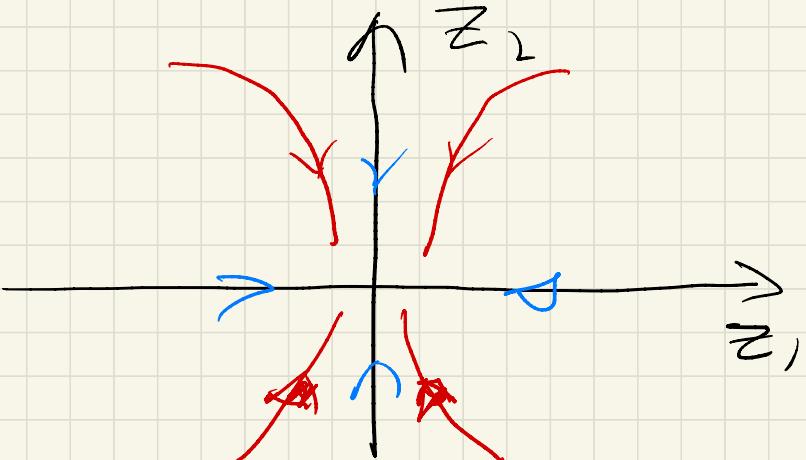
$$z_i(t) = e^{\lambda_i t} z_i(0)$$

$$\begin{matrix} \pi \\ \rightarrow \end{matrix}$$



CANNOT cross eigendirections in the phase plane [separatrix]

Q2)  $\lambda_1, \lambda_2 < 0$  NODE



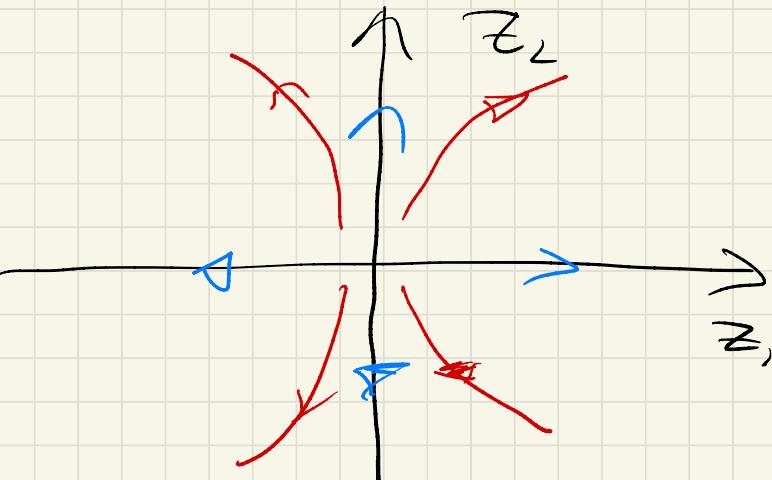
*stable*  
(stable NODE)

Q4)  $\lambda_1 = 0, \lambda_2 < 0$ ;  $z_1(0) = z_2(0) = 0$

$$\dot{z}_1 = 0; \dot{z}_2 = -2z_2$$

*borderline*

Q3)  $\lambda_1, \lambda_2 > 0$



*source*

(unstable NODE)

$$z_1(t) = 0; z_2(t) = e^{-2t} z_2(0)$$

$$z_1(t) = 0; z_2(t) = e^{-2t} z_2(0)$$



②

$$\text{Ex: } \dot{z}_1 = -z_1^3 \quad (\text{or} \quad \dot{z}_1 = +z_1^3)$$

$$\dot{z}_2 = -2z_2 - z_2^5$$