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## Lecture 7

02/06/23

Last time:

• Phase portraits of 2nd order LTI systems

1° Real e-values  $\lambda_1 \neq \lambda_2$

①)  $\lambda_1 > 0, \lambda_2 < 0$ ; saddle

②)  $\lambda_1 < 0, \lambda_2 < 0$ ; stable node

③)  $\lambda_1 > 0, \lambda_2 > 0$ ; unstable node

Today:

2°  $\lambda = \alpha \pm j\beta$ ;  $\beta \neq 0$  [ stable or unstable  
screw  
 $\alpha < 0$        $\alpha > 0$  ]

center:  $\alpha = 0$

3° Jordan forms: ex:  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  is a Jordan block

2<sup>o</sup> E-values of the matrix A : C.C.

$$\lambda(A) = \alpha + j\beta ; \alpha = \operatorname{Re}(\lambda(A)) ; \beta = \operatorname{Im}(\lambda(A))$$

$$\bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} ; \text{Polar coordinates:}$$

$$z_1 = r \cdot \cos(\theta)$$

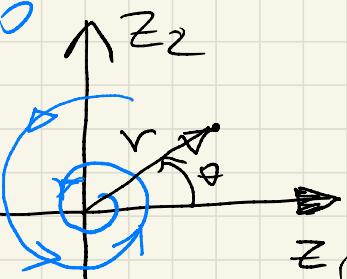
$$z_2 = r \cdot \sin(\theta)$$

Can show:

$$\dot{r} = \alpha \cdot r \rightarrow r(t) = e^{\alpha t} r(0)$$

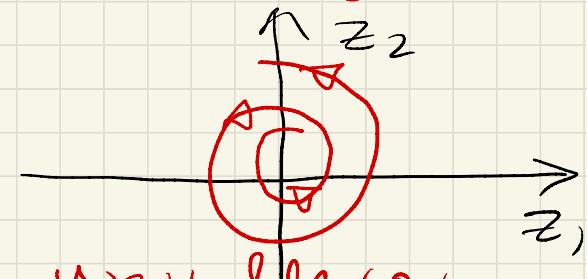
$$\dot{\theta} = \beta \rightarrow \theta(t) = \theta(0) + \beta \cdot t$$

2a)  $\alpha < 0$



stable focus

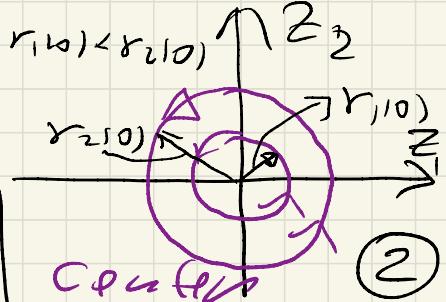
2b)  $\alpha > 0$



unstable focus

Hoch-relevant  
(fragile)  
✓ esp. case

2c)  $\alpha = 0$



center

②

3° An example of a Jordan form

$$\bar{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

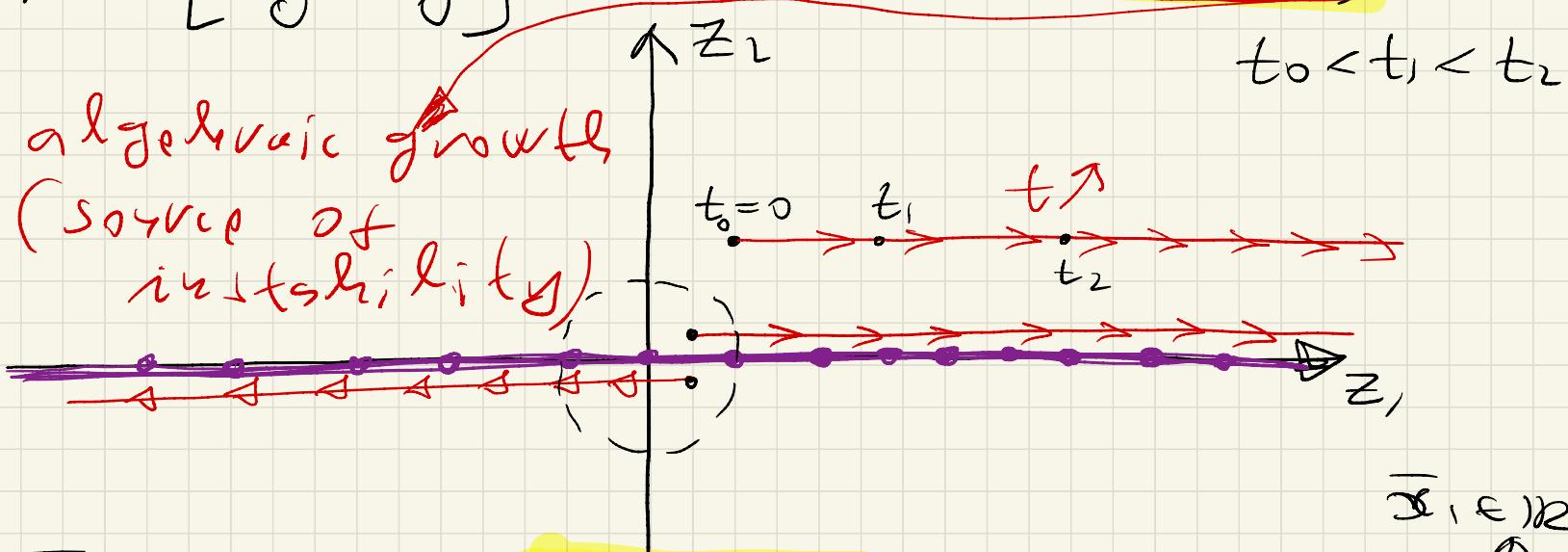
Doubly integrable;  $\lambda = 0$   
 $\ddot{y} = 0$

$$\dot{z}_1 = z_2 \rightarrow z_1(t) = z_1(0) + z_2(0) \cdot t$$
$$\dot{z}_2 = 0 \rightarrow z_2(t) = z_2(0)$$

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

algebraic growth

(source of instability)



$\bar{x} = 0$  is a UNSTABLE eq. point  
(as well as any other eq. point  $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$ )

$$\bar{A} = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix}; \quad \lambda \neq 0$$

$$e^{\bar{A}t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \text{unique eq. point}$$

$$\lambda < 0 \Rightarrow \lim_{t \rightarrow \infty} (x(t)) = 0$$

(is spite of transient growth  
i.e., non-monotonic time  
dependence)

$$\text{e.g. } \lambda = -2$$

$$e^{\bar{A}t} = \begin{bmatrix} e^{-2t} & t \cdot e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}; z(t) = e^{\bar{A}t} \cdot z(0)$$

$$z_1(t) = e^{-2t} \cdot z_1(0) + t \cdot e^{-2t} \cdot z_2(0)$$

$$z_2(t) = e^{-2t} \cdot z_2(0)$$

$$\dot{z}_1 = -2z_1 + z_2 \rightarrow z_1(s) = \frac{1}{s+2} \cdot z_1(0) + \frac{1}{(s+2)^2} z_2(0)$$

$$\dot{z}_2 = -2z_2 \rightarrow z_2(s) = \frac{1}{s+2} z_2(0)$$

5

Aside: Compare with:

$$\bar{A} = \begin{bmatrix} -1 & K \\ 0 & -2 \end{bmatrix} \quad \begin{aligned} \dot{z}_1 &= -z_1 + K z_2 \\ \dot{z}_2 &= -2 z_2 \end{aligned}$$

$$z_2(t) = e^{-2t} \cdot z_2(0)$$

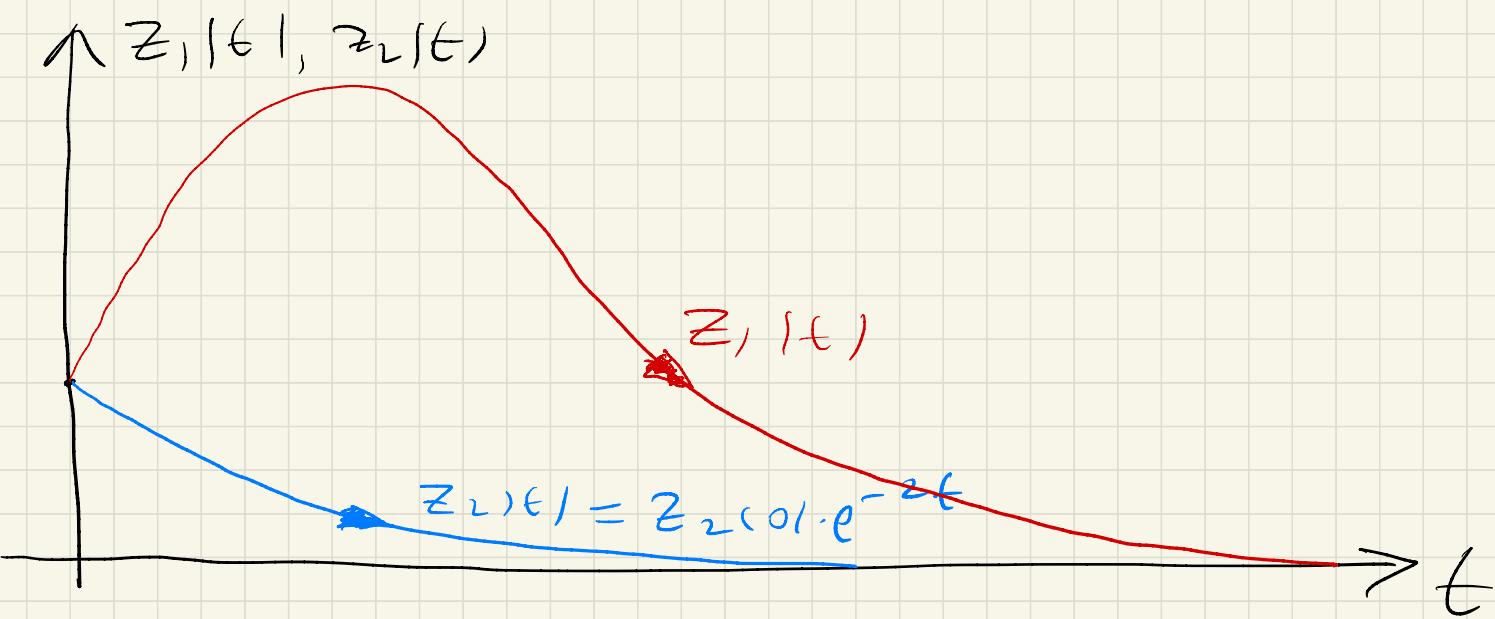
$$z_1(t) = e^{-t} \cdot z_1(0) + K \cdot \left( e^{-t} - e^{-2t} \right) \cdot z_2(0)$$

check signs

Compare with

$$z_1(t) = e^{-2t} z_1(0) + t \cdot e^{-2t} z_2(0)$$

→ Jordan form



(7)

Q: What does this have to do with phase portraits of nonlinear systems?

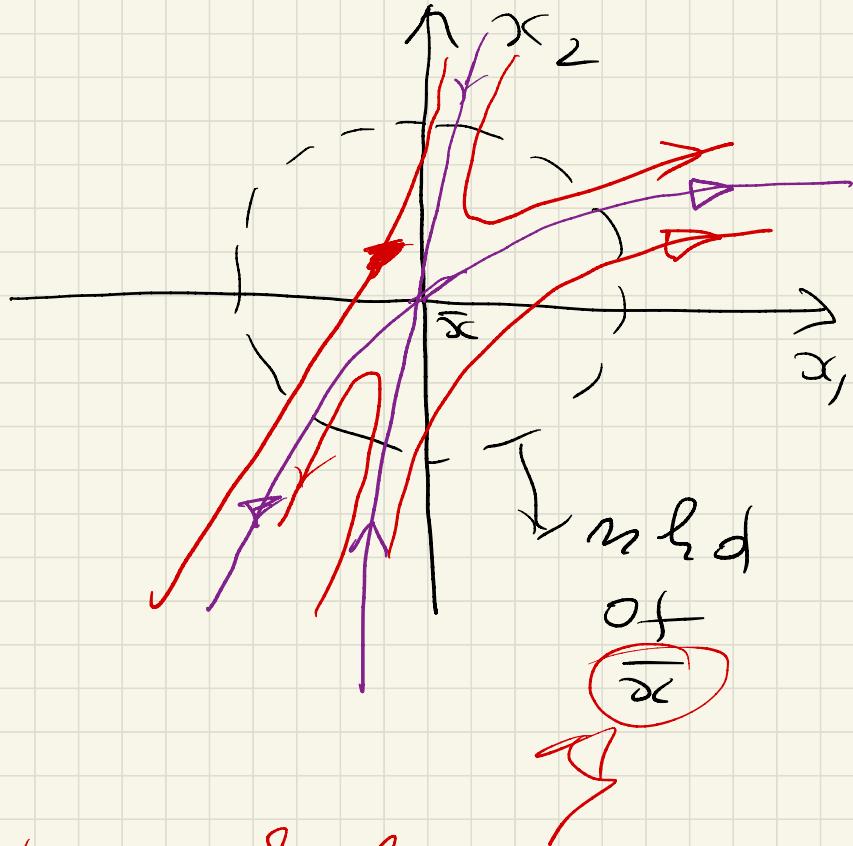
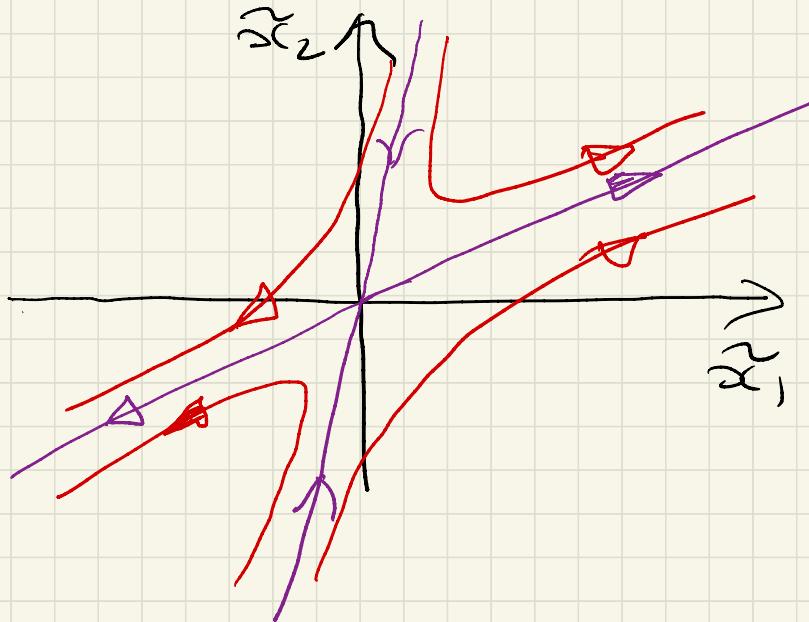
$$\dot{x} = f(x) \text{ v.s. } \dot{\tilde{x}} = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}} \cdot \tilde{x}$$

A: Hartman - Grobman Theorem for HYPERBOLIC eq. points

(i.e. Linearization around  $\bar{x}$  DOES NOT have  $\lambda$ -values on  $jw$ -axis)

Real part of  $\lambda$ -values of  $A = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}}$  stability boundary is NOT zero

→ Allows us to relate trajectory of  $\dot{x} = f(x)$  &  $\dot{\tilde{x}} = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}} \cdot \tilde{x}$  locally (around  $\bar{x}$ ) via "nice" coord. transf. (B)



Hyperbolic eq.  
point



HB Theory: If  $\bar{x}$  is a hyperbolic eq. point of  $\dot{x} = f(x)$ , with  $x(t) \in \mathbb{R}$  then there is a cts mapping  $\bar{x}$  with cts inverse [HOMEOMORPHISM]

described is the cts of  $\bar{x}$  that maps trajectories of  $\dot{x} = f(x)$  to trajectories of  $\dot{\bar{x}} = \frac{\partial f}{\partial x} \bar{x}$ .

Key: requirement on the slope of eigenvalues on jw-axis CANNOT be relaxed.

$$\text{Ex: } \begin{cases} \ddot{x}_1 = -\omega_2 + \eta x_1 (\omega_1^2 + \omega_2^2) \\ \ddot{x}_2 = +\omega_1 + \eta x_2 (\omega_1^2 + \omega_2^2) \end{cases}$$

$$\text{linearization: } \begin{cases} \ddot{x}_1 = +\omega_1 \\ \ddot{x}_2 = +\omega_2 \end{cases}$$

linearization

$$\overline{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}; \text{ eq. point:}$$

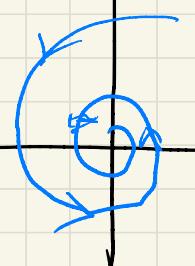
$$\det(\sigma I - A) = \sigma^2 + 1 \Rightarrow \lambda = \pm j$$

Harmonic oscillator

In polar coordinates

$$\dot{r} = a \cdot r^3 \quad a) \quad \eta < 0$$

$$\dot{\theta} = 1$$



Stable focus

$\omega_1$

unstable

focus



b)  $\eta > 0$



unstable focus

