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# Lecture 8

02/08/23

Last time:

- Hartman - Grobman Thm

(relation between trajectories of  $\dot{x} = f(x)$   
and  $\dot{\tilde{x}} = \frac{\partial f}{\partial x}|_{x=\tilde{x}}$   $\tilde{x}$  around hyperbolic  
eq. points;  $\rightarrow$  NO jw-axis e-values)

Today:

- Bifurcations Thm [absence]  $\rightarrow$  of periodic orbits
- Positively invariant sets
- Poincaré-Bendixson Thm [presence]

for  $\dot{x} = f(x)$ ;  $x(t) \in \mathbb{S}^2$

"Planar" systems [2nd order;  $\dot{x}(t) \in \mathbb{R}^2$ ]

$$\dot{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; \quad x_i(t) \in \mathbb{R}$$

$$\boxed{\ddot{x} = f(x)}$$

(LT)

$$\left. \begin{array}{l} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{array} \right\}$$

$$f_i: \mathbb{R}^2 \rightarrow \mathbb{R}$$

*no explicit time dependence in f*

Theorem [Bendixson's criterion for absence of periodic orbits]:

If  $\operatorname{div} f = \nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$  is NOT

identically equal to zero AND it doesn't change sign in a simply connected domain  $\mathcal{D}$  [domain w/o holes] in  $\mathbb{R}^2$  then there are No Periodic orbits

Note:  $\operatorname{div} f = \nabla \cdot f = \operatorname{trace} \left\{ \frac{\partial f}{\partial x} \right\}$

trace of Jacobian

Ex: LTI system with  $n=2$

$$f(x) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$

$\frac{\partial f}{\partial x}$

$$\operatorname{div} f = \operatorname{trace}(A) = a_{11} + a_{22}$$

coffs +  
(real  
numbers)

$\Rightarrow \operatorname{div} f$ : CAN NOT change sign!!!

$$\Theta: a_{11} + a_{22} = 0$$

If  $N \neq T \Rightarrow$  No periodic orbits ③

Periodic orbits can exist if  
 $\sigma_{11} = -\sigma_{22}$  (e.g.  $\sigma_{11} = \sigma_{22} = 0$ )

$$A = \begin{bmatrix} 0 & \sigma_{12} \\ \sigma_{21} & 0 \end{bmatrix}$$

If  $\sigma_{11} + \sigma_{22} \neq 0 \Rightarrow$   
 No Periodic orbits

Analyzing beyond Appendix:

$$\det(SI - A) = S^2 - \sigma_{12} \cdot \sigma_{21}$$

This is ACC  
 (Just This  
 tells us!!!)

If  $\text{sign}(\sigma_{12}) = \text{sign}(\sigma_{21}) \neq 0 \Rightarrow$

No periodic orbits [ saddle:  $\lambda_{12} = \pm \sqrt{\sigma_{12}\sigma_{21}}$  ]

If  $\sigma_{12} \cdot \sigma_{21} < 0$  also we have  $\sigma_{12} = \sigma_{21} = 0$

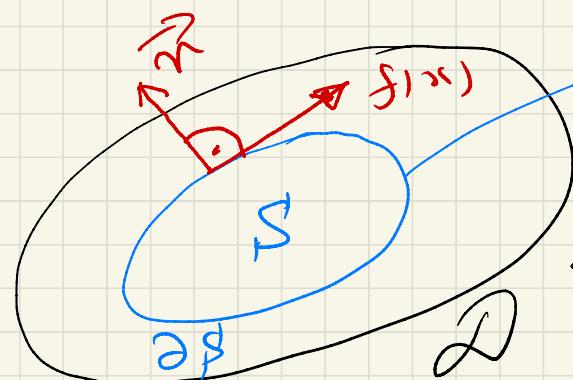
$$\det(SI - A) = S^2 + \sigma_{12} \cdot \sigma_{21}$$

$\lambda_{12} = \pm j \sqrt{|\sigma_{12}\sigma_{21}|} \Rightarrow$  Instinctively many periodic orbits (center)

Proof: as immediate consequence of divergence Thm

Proof by  $\rightarrow \leftarrow$  [contradiction]

Assume that there is a periodic orbit and show that under the conditions of the Thm this is NOT possible.



Periodic orbit is  $L$   
 $f(x)$ : tangential to periodic orbit  
 $\vec{n}$ : outward normal

$$\dot{x} = f(x) ; [f(x)]^T \cdot \vec{n} = 0$$

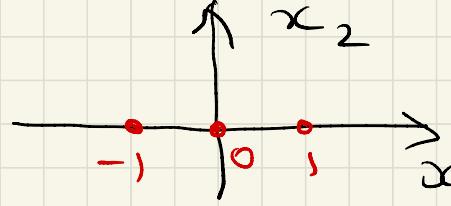
↑ div. Thm

$$0 = \int_{\partial S} [f(x)]^T \cdot \vec{n} \cdot dL = \iint_S \text{div}(f) dS$$

$$\begin{aligned} \text{Ex: } \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\underline{\alpha} \cdot x_2 + x_1 - x_1^3 + x_1^2 x_2 =: f_2(x_1, x_2) \end{aligned}$$

$\alpha > 0$ : positive parameter

Eig. points:  $\begin{cases} 0 = \bar{x}_2 \\ 0 = \bar{x}_1(1) - \bar{x}_1^2 \end{cases} \Rightarrow$  3 Eig. points

$$\bar{x} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$


$\text{div } f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 - \alpha$

Need to examine signs of

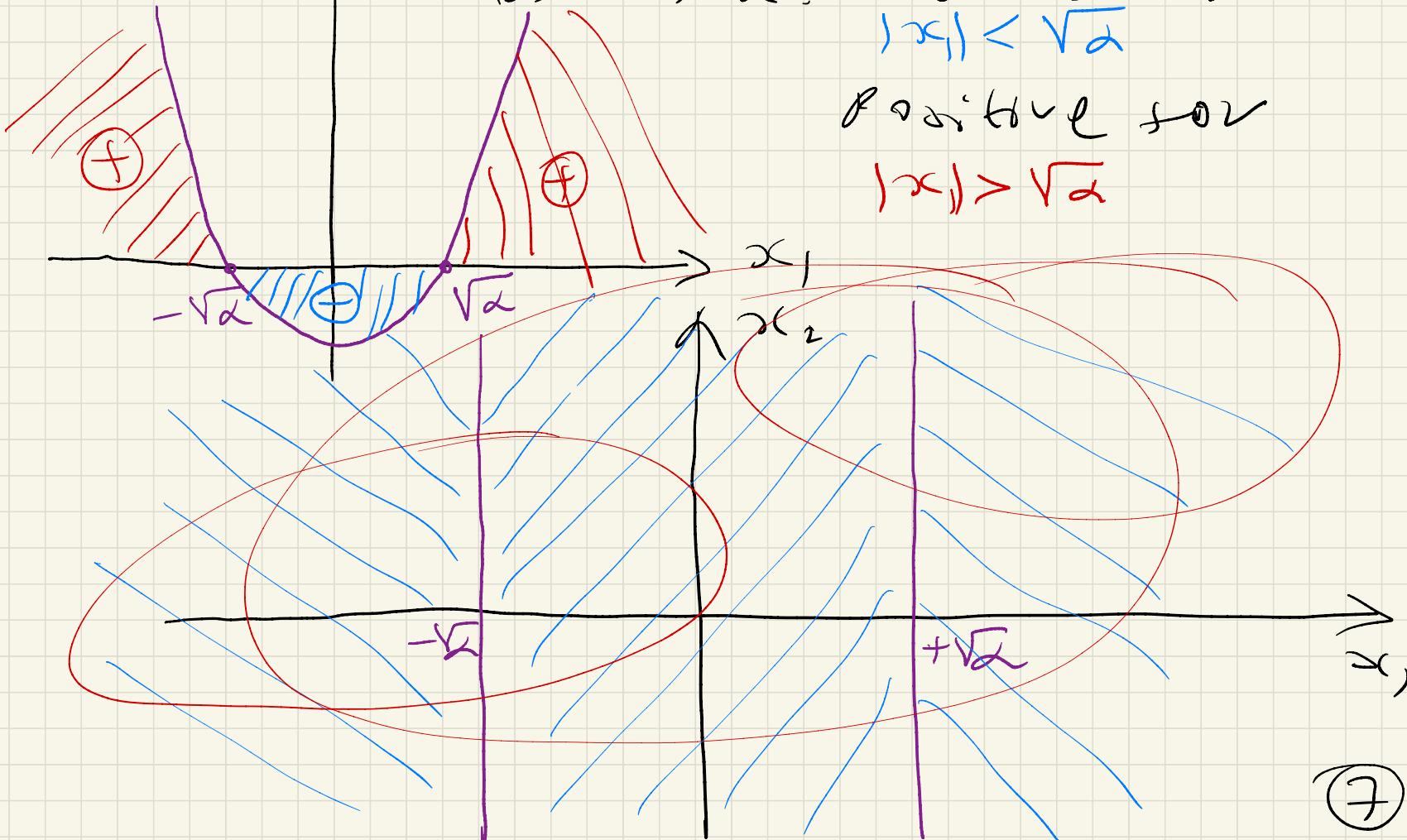
$$x_1^2 - \alpha$$

$$\operatorname{div}(f) = x_1^2 - \alpha : \text{Negative for } |x_1| < \sqrt{\alpha}$$

$$|x_1| < \sqrt{\alpha}$$

Positive for

$$|x_1| > \sqrt{\alpha}$$



(7)

No periodic orbits for

$$x_1 \in (-\alpha, -\sqrt{\alpha}) ; x_1 \in (-\sqrt{\alpha}, \sqrt{\alpha})$$

$$x_1 \in (\sqrt{\alpha}, +\infty)$$

What we can conclude based on Bendixson Thm.

RED circled regions on the previous page:

CANNOT say anything

Bendixson's Thm is inconclusive

CAN'T conclude anything for

$x_1 \in (a, b)$  if either  $-\sqrt{\alpha}$  or  $\sqrt{\alpha}$  belongs to  $(a, b)$ .

IF Index Theory can provide insight  
additional ⑧

## Positively Invariant Sets

$\dot{x} = f(x)$  with  $x \in \mathbb{R}^n$ ,  $x(t_0) = x_0 \in \mathbb{R}^n$

Solution:  $x(t) = \phi(t, x_0)$  @ time  $t$   
starting from  $x_0$  @  $t=t_0$ .

Def: A set  $M \subset \mathbb{R}^n$  is positively invariant if  $x_0 \in M \Rightarrow \phi(t, x_0) \in M$  for all  $t \geq t_0$

Special cases (e.g. points; <sup>closed</sup> periodic orbits;  
inside of a periodic orbit for  
planar systems; region of attractions  
of e.g. point.)