

---

---

---

---

---

---



# Lecture 9

02/13/23

Last time:

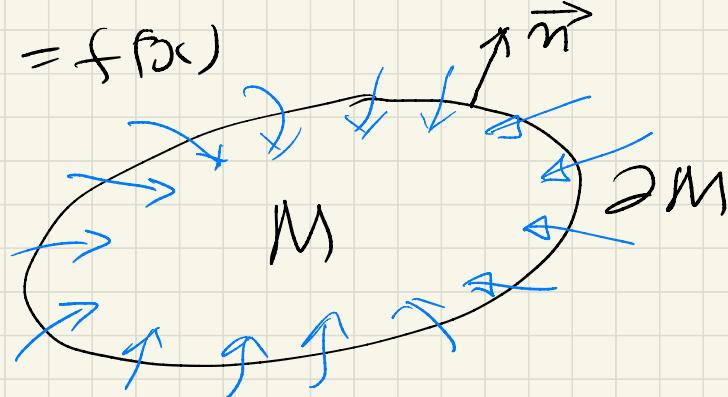
- Bendixson's Thm (absence of periodic orbits for 2nd order - "planar" =  $T^1$  systems)
- Positively invariant sets (definition)  
[if you start in the set  $M$  @  $t_0$ , you stay in  $M$  for all  $t \geq t_0$ ]

Today:

- . More on positively invariant sets
- . Poincaré-Bendixson's Thm (existence of periodic orbits for 2nd order systems  $\dot{x} = f(x)$ )

Set  $M$  with closed boundary  $\partial M$

$$\dot{x} = f(x)$$



$$[f(x)]^T \cdot \vec{n} \leq 0$$

along  $\partial M$

$$\langle f(x), \vec{n} \rangle \leq 0$$

inner product between  
 $f(x)$  and  $\vec{n}$

Note: Positive invariance is also useful  
tool for stability analysis of  
e.g. points of  $\dot{x} = f(x)$

Level sets of Lyapunov functions are  
positively invariant (more later) (2)

## Predator / Prey Model

(population of)

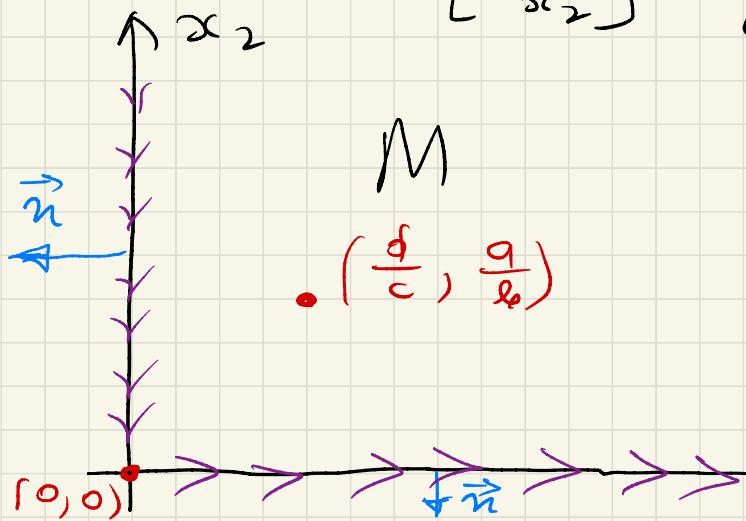
$$\dot{x}_1 = (a - b x_2) x_1 : \text{prey} \quad [\text{e.g. zebras}]$$

$$\dot{x}_2 = (c x_1 - d) x_2 : \text{predator} \quad [\text{e.g. lions}]$$

$a, b, c, d$  : positive parameters

$x_1(t), x_2(t) \in \mathbb{R}^2$  : scalar states

Ex. points :  $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} d/c \\ a/b \end{bmatrix} \end{cases}$



Note: model should reflect the fact that predators & prey are non-negative & coexisting [i.e. we should have positive initial value of the 1st quadrant]

(3)

$$\begin{aligned} f_1(x_1, x_2) &= (a - b x_2) x_1 \\ f_2(x_1, x_2) &= (c x_1 - d) x_2 \end{aligned} \Rightarrow \frac{\partial f}{\partial x} \Big|_{\bar{x}=0} = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$$

*saddle*

$x_1 = 0$  (vertical axis)  $\Rightarrow f_1 \equiv 0 ; f_2 = -d x_2$   
 i.e.  $\dot{x}_1 = 0 ; \dot{x}_2 = -d x_2 \Rightarrow x_2(t) = e^{-dt} \cdot x_2(0)$

Population of lions decays exponentially to zero (with rate  $d$ ) if  $H_2$  zero

$x_2 = 0$  (horizontal axis);  $f_1 = a x_1 ; f_2 \equiv 0$

$\dot{x}_1 = a x_1 \Rightarrow x_1(t) = e^{at} x_1(0)$

$\dot{x}_2 = 0 \Rightarrow x_2(t) \equiv 0$  ( $H_1$  LHS)

Population of zebras grows exp. with rate  $a$  if  $H_1$  LOTS

We showed that  $[f_1(x)]^T \cdot \vec{n} = 0$  along  $\partial M$ : boundary of the 1st quadrant  $\Rightarrow$

M is a positively invariant set

$\hookrightarrow$  1st quadrant

$$M := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 ; x_1 \geq 0 ; x_2 \geq 0 \right\}$$

Predator/Prey Model is an example of positive (or more generally monotone)

systems

LT system is positive

For  $\dot{x} = Ax$

A: Mertzler matrix; i.e.  $a_{ij} \geq 0$ ; for  $i \neq j$  (5)

$$A = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$$

$$\dot{x}_i = -2x_i + x_{i+1} + x_{i-1} =$$

$$= -2 \cdot \left[ x_i - \frac{x_{i+1} + x_{i-1}}{2} \right]$$

$$\bar{x} = \frac{1}{n} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

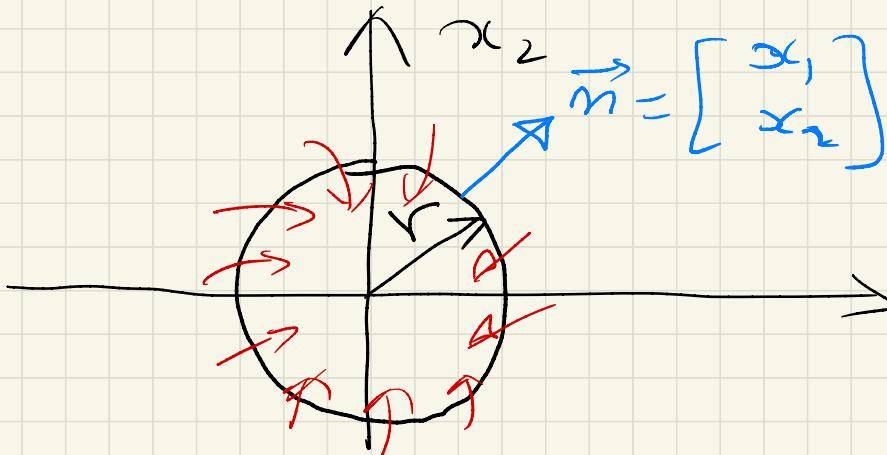
$$A \bar{x} = 0 \Rightarrow \bar{x} = 0 ; \bar{x} = \underline{1}$$

$$\text{Ex: } \begin{cases} \dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{cases} =: f_1(x_1, x_2)$$

Show that a circle of radius  $r$

$$B_r = \{ x \in \mathbb{R}^2; x_1^2 + x_2^2 \leq r^2 \}$$

is positively invariant for LARGE ENOUGH values of  $r$ .



Need to show  
that there is

$$\bar{r} \text{ s.t. } r \geq \bar{r}$$

$$x_1 (f_1(x))^\top \cdot \vec{n} \leq 0$$

(g)

$$[f(\alpha)]^T \cdot \vec{n} = f_1 \cdot \underline{\alpha_1} + f_2 \cdot \underline{\alpha_2} \Rightarrow \text{for circle along its boundary}$$

$$= [\underline{\alpha_1 + \alpha_2 - \alpha_1 \{ \alpha_1^2 + \alpha_2^2 \}}] \alpha_1 + [\underline{-2\alpha_1 + \alpha_2 - \alpha_2 \{ \alpha_1^2 + \alpha_2^2 \}}] \alpha_2$$

$$= \cancel{\alpha_1^2} + \alpha_1 \alpha_2 - \underline{\alpha_1^2 \{ \alpha_1^2 + \alpha_2^2 \}} - 2\alpha_1 \alpha_2 + \cancel{\alpha_2^2} - \cancel{\alpha_2^2 \{ \alpha_1^2 + \alpha_2^2 \}}$$

$$= (\alpha_1^2 + \alpha_2^2) [1 - \{ \alpha_1^2 + \alpha_2^2 \}] - [\alpha_1 \alpha_2]$$

of rotational source of torque

$$\leq \underbrace{\alpha_1^2 + \alpha_2^2}_{r^2} [1 - \underbrace{\alpha_1^2 + \alpha_2^2}_{r^2}] + \frac{1}{2} \underbrace{\{ \alpha_1^2 + \alpha_2^2 \}}_{r^2} =$$

sign-isol.

$$= r^2 \left[ \frac{3}{2} - r^2 \right] \leq 0 \Rightarrow$$

$$\rightarrow r^2 \geq \frac{3}{2} \Rightarrow [f(\alpha)]^T \cdot \vec{n} \leq 0$$

(b)

Note:  $(x_1 \pm x_2)^2 \geq 0$

$$x_1^2 \pm 2x_1 x_2 + x_2^2 \geq 0$$

$$x_1^2 + x_2^2 \geq -2x_1 x_2$$

$$x_1 x_2 \leq \frac{1}{2} \cdot (x_1^2 + x_2^2)$$

∴

Thus,  $\sigma_B r$  is  $\text{+}$  invariant if

$$r \geq \sqrt{\frac{3}{2}}$$

PB Thm [basic version] existence of periodic orbits for  $\dot{z} = f(z)$  with  $z(t) \in \mathbb{H}^2$

Let  $M$  be a compact [i.e. closed & bounded] set in  $\mathbb{H}^2$  which is positively invariant for  $\dot{z} = f(z)$  with  $z(t) \in \mathbb{H}^2$ .

If  $M$  DOES NOT contain an eq. point then it contains a periodic orbit.

Refined version: Requirement on "NO eq. points" can be relaxed to:

//  $M$  can have one eq. point which is either on unstable focus or a unstable node".  $\lambda = \alpha \pm i\beta$ ;  $\alpha > 0$   
 $\lambda_1, \lambda_2 > 0$ , real

In the previous example,

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{\partial f}{\partial x} \Big|_{x=0} = \bar{x} = 0$$

$\det(sI - A) =$

$$\begin{vmatrix} s-1 & -1 \\ 2 & s-1 \end{vmatrix} =$$

$$= (s-1)^2 + 2 =$$

$$= s^2 - 2s + 3$$

unstable

positive real parts  
of c.c. eigenvalues  
 $\Rightarrow$  unstable focus.

$$s_1, s_2 = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm j2\sqrt{2}}{2} = 1 \pm j\sqrt{2}$$

$\Rightarrow \exists$  a periodic orbit in  $\mathbb{R}^2$  for  $r \geq \sqrt{\frac{3}{2}}$ . (23)