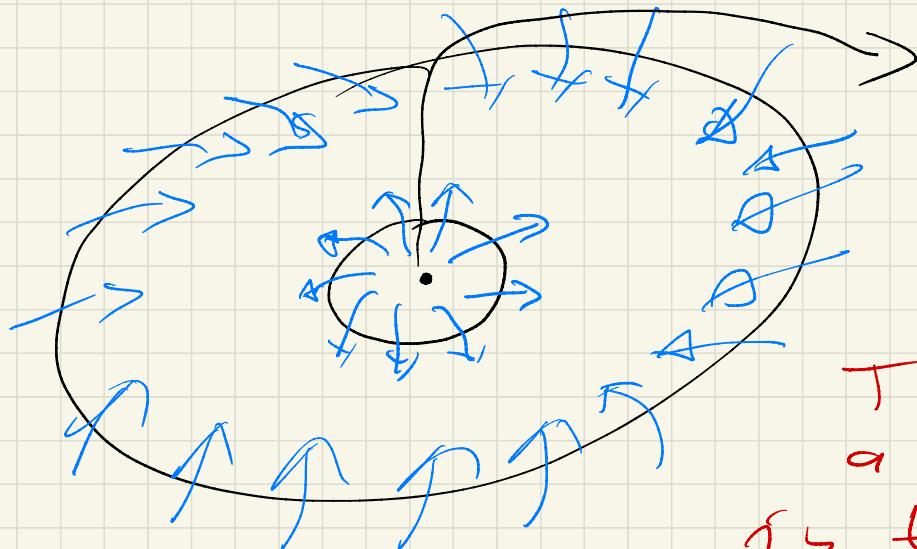



Lecture 10

02/15/23

Last time:

- Positively invariant sets (Examples)
- PB Thm (presence of ^{sufficient} ^{conditions} periodic orbits for planar time-invariant systems)



Unstable Node
or Focus

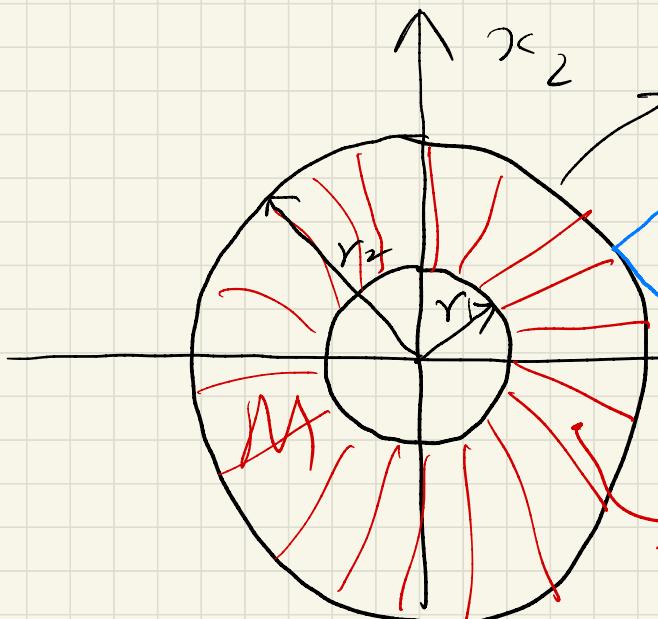
(e.g. is the ^{LiBr} example)

Then there is
a periodic orbit
in the \oplus invariant
set.

Ex.: Harmonic oscillator; $K = m$:



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Trajectory with \vec{n}
vector \vec{n} determined

$$f(\vec{x}) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$x_1, [f(x)]^T \vec{n} = 0$$

⊕ invariant set
(No eq. points)

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}; \vec{n} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \left| \begin{array}{l} \text{PB} \Rightarrow \text{② least one} \\ \text{periodic orbit if } M \text{ ②} \\ (\text{shaded region}) \end{array} \right.$$

Yet another refinement: index theory
(please see Mazzat's notes)

Hopf Bifurcations

So far: 1st order "phenomena"

a) $\dot{x} = \lambda + x^2$: fold

b) $\dot{x} = \lambda x + x^2$: transcritical

c) $\dot{x} = \lambda x + x^3$: pitchfork

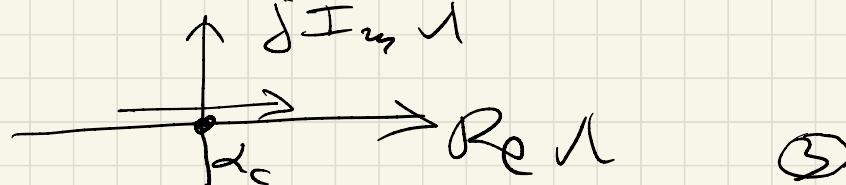
Everywhere they appear in higher order systems

Common theme: Linearization $\text{④ } \lambda = \lambda_c$ critical λ

Was NOT INFORMATIVE

$$\bar{x} = x(\lambda_c)$$

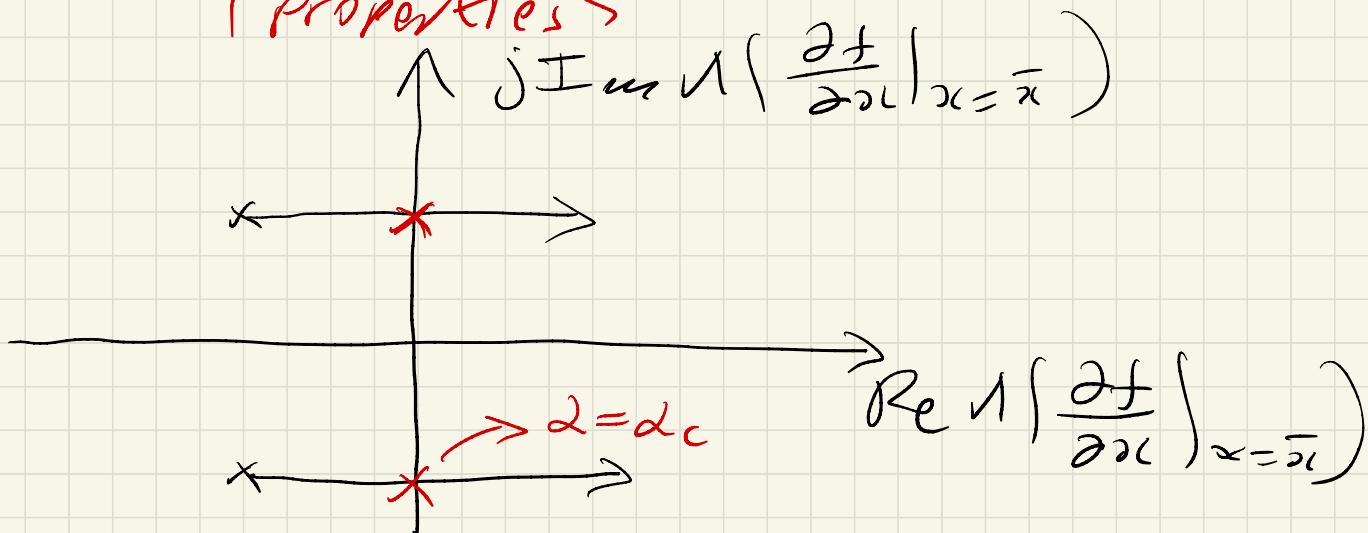
$$\frac{\partial f}{\partial x} \left|_{\begin{array}{l} \lambda = \lambda_c \\ \bar{x} = x(\lambda_c) \end{array}} \right. = 0$$



\hookrightarrow 1st order bifurcations: change in equilibrium landscape and/or in their stability properties

Hopf bifurcations: appear in 2nd order

[Interplay between limit cycles and stability of $\bar{x} = 0$]
~~Properties~~



Hopf bifurcations

a) supercritical



b) subcritical



$$\text{a) Ex: } \dot{x}_1 = x_1 \cdot [\alpha - x_1^2 - x_2^2] - \alpha x_2$$

$$\dot{x}_2 = x_2 \cdot [\alpha - x_1^2 - x_2^2] + \alpha x_1$$

Polar coordinates: $x_1 = r \cdot \cos(\theta)$

$$x_2 = r \cdot \sin(\theta)$$

$\dot{r} = \alpha r - r^3 \rightarrow$ looks like a supercritical pitchfork bifurcation

$$\dot{\theta} = 1$$

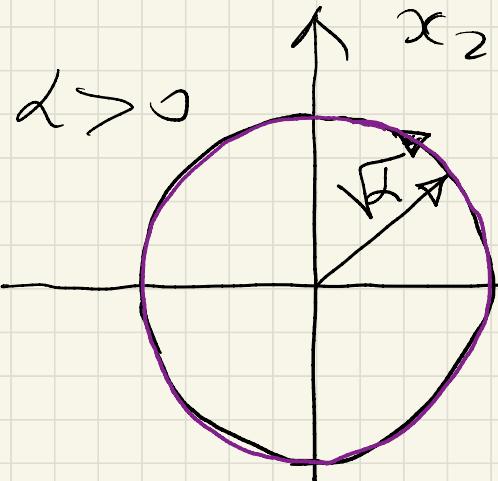
$$\bar{r} = 0 ; \bar{r}^2 = \alpha$$

$$\dot{r} = 0 = \bar{r}(\alpha - \bar{r}^2) = 0 \Rightarrow$$

$$\bar{r} = \begin{cases} 0 \\ \sqrt{\alpha} ; \alpha > 0 \end{cases}$$

(S)

Note: $\bar{r} = 0$: Eq. point
 For $d > 0$, $\bar{r} = \sqrt{d}$ (periodic orbit) [isolated]
 $\dot{\theta} = 1$] Limit cycle



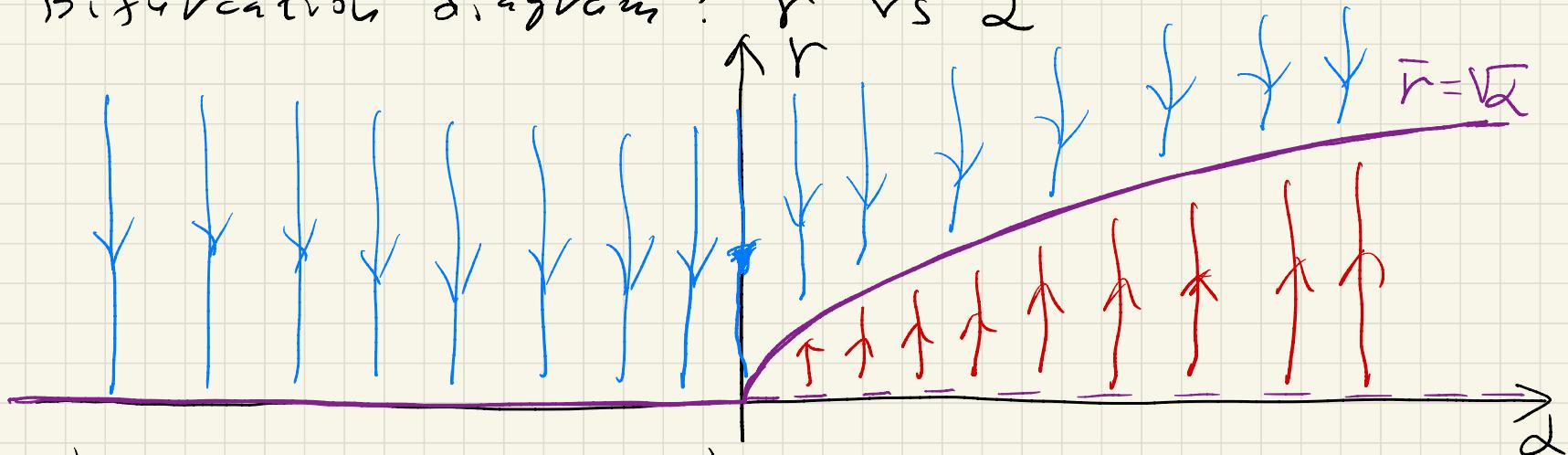
$\dot{r} = f_r(r, d)$ Linearization:
 (Jacobian)

$$d > 0, \frac{\partial f_r}{\partial r} = d - 3r^2$$

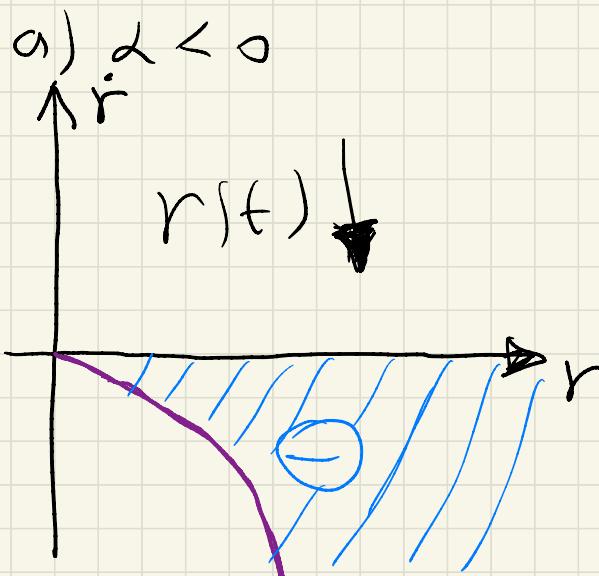
$$\left. \frac{\partial f_r}{\partial r} \right|_{r=\bar{r}} = \begin{cases} d & ; \bar{r} = 0 \\ -2d & ; \bar{r} = \sqrt{d} \end{cases}$$

- $d < 0 \Rightarrow \bar{r} = 0$ is locally asymptotically stable
- $d > 0 \Rightarrow \bar{r} = 0$ is UNSTABLE
- $d = 0 \Rightarrow$ linearization is NOT INFORMATIVE (S)

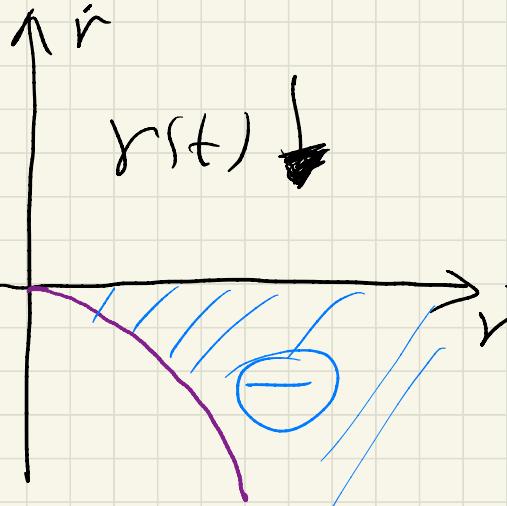
Bifurcation diagram: r vs α



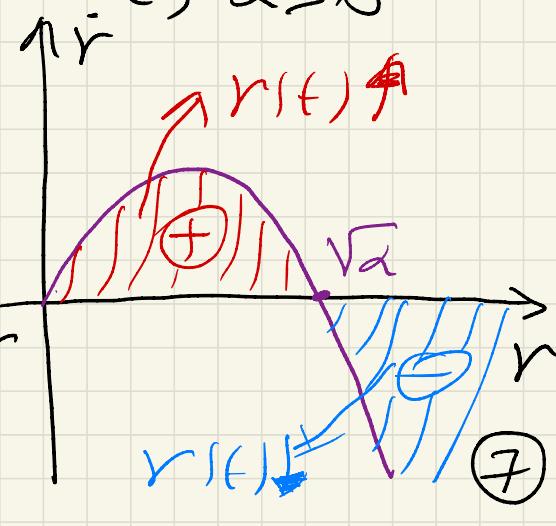
a) $\alpha < 0$



b) $\alpha = 0$



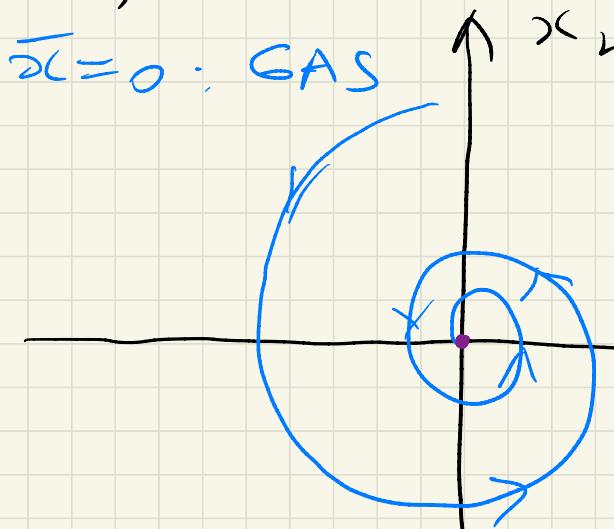
c) $\alpha > 0$



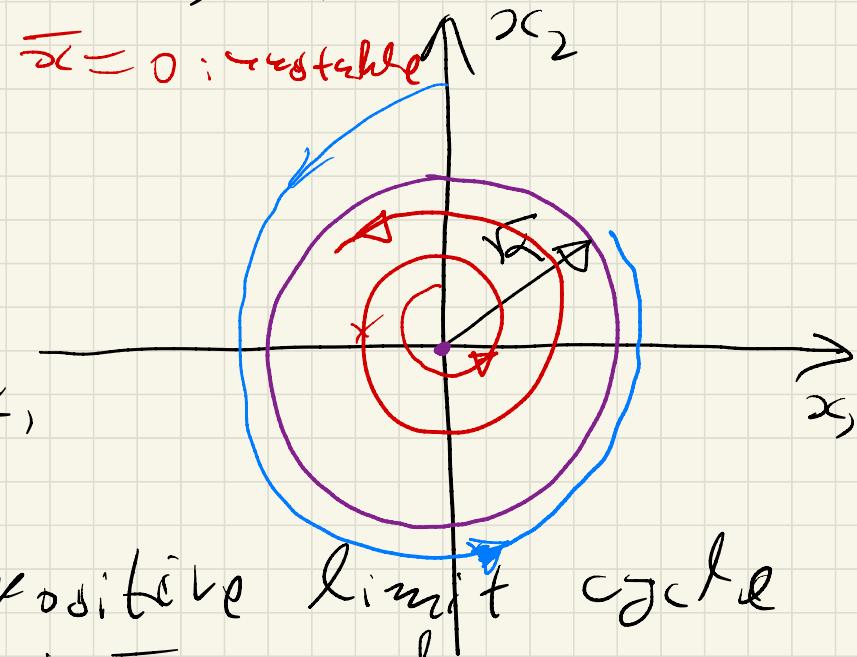
(7)

(α_1, α_2) -plane :

a) $\alpha \leq 0$



b) $\alpha > 0$



Summary: As α becomes positive limit cycle of radius $\sqrt{\alpha}$ emerges; $\bar{\alpha} = 0$ becomes unstable; and all trajectories with $\alpha(0) \neq 0$ converge to the limit cycle ⑧

6) Subcritical Hopf See local for

$$\begin{cases} \dot{r} = \alpha r + r^3 - r^5 \\ \dot{\phi} = 1 \end{cases}$$

$$\dot{r} = 0 \Rightarrow \bar{r} \cdot (\alpha + \bar{r}^2 - \bar{r}^4) = 0 \quad \bar{r} = \bar{r}^2$$

$$\bar{r} = 0 \text{ or } \bar{r}^2 - \bar{r} - \alpha = 0 \quad \left\{ \begin{array}{l} \text{For values of} \\ \alpha \text{ that give } \bar{r} \in \mathbb{R}_+ \end{array} \right.$$

$$\bar{r}_{1,2} = \frac{1 \pm \sqrt{1+4\alpha}}{2}$$

$$\bar{r} = \sqrt{\frac{1}{2} \left\{ \left(\pm \sqrt{1+4\alpha} \right) \right\}}$$

$$\alpha = -\frac{1}{4} \quad \& \quad \alpha = 0$$

INTERESTING Values of α

$$\alpha \in \left[-\frac{1}{4}, 0 \right]$$