


Lecture 11

02/22/23

b) Subcritical Hopf

$$\dot{r} = \lambda \cdot r + r^3 - r^5 = r (\lambda + r^2 - r^4)$$

$$\dot{\theta} = 1$$

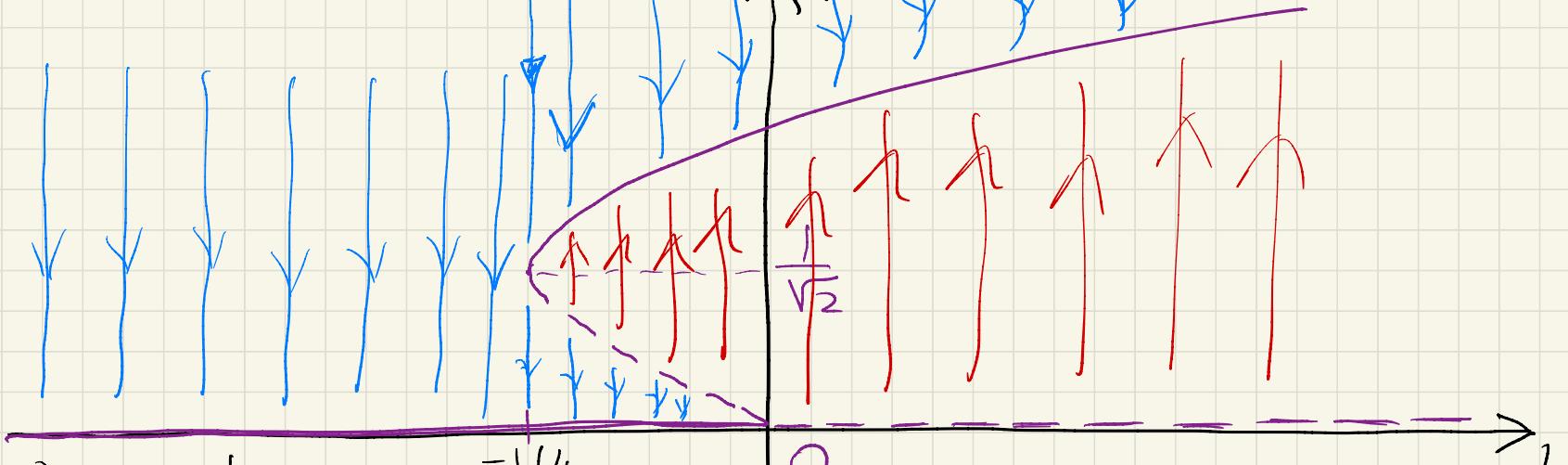
$$\dot{r} = 0 \Rightarrow \bar{r} = 0 \quad \text{or} \quad \bar{r}^2 - \bar{r} - \lambda = 0$$
$$\bar{r} := r^2$$

$$\bar{r} = \sqrt{\frac{1}{2} \cdot \left[1 + \sqrt{1 + 4\lambda} \right]}$$

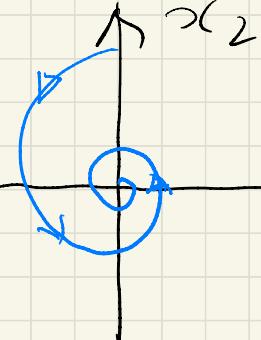
$\lambda > 0$
 \oplus
 $\Rightarrow \lambda \in [-\frac{1}{4}, 0]$

$\lambda = -\frac{1}{4}$ & $\lambda = 0$: values of λ for which limit cycles emerge or vanish

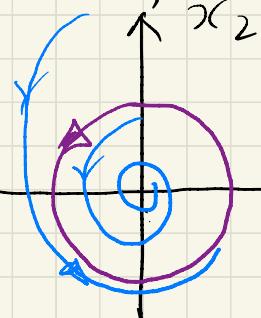
Bifurcation diagrams (in (λ, r) -plane)



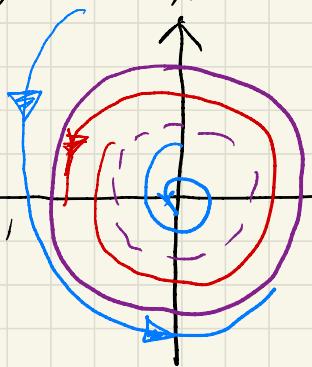
a) $\lambda < -\frac{1}{4}$



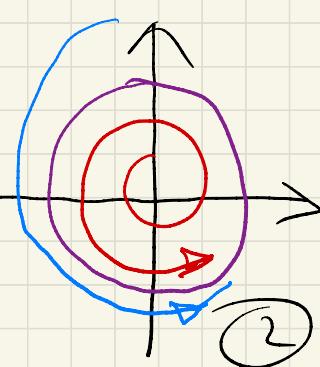
b) $\lambda = -\frac{1}{4}$



c) $\lambda \in (-\frac{1}{4}, 0)$



d) $\lambda \geq 0$



Note : For $\lambda = -\frac{1}{4}$ $\dot{r} = -r \cdot \left(r - \frac{\sqrt{2}}{2}\right)^2$

For $\lambda \in (-\frac{1}{4}, 0)$ $\dot{r} = -r \cdot (r - \bar{r}_1)(r - \bar{r}_2)$
 $\bar{r}_1 > 0 ; \bar{r}_2 > 0$

For $\lambda \geq 0$ $\dot{r} = -r(r - \bar{r}_1) \cdot (r + K)$

For $\lambda < -\frac{1}{4}$; $\dot{r} = -r \cdot \underbrace{\left(r^4 - r^2 - \lambda\right)}_{> 0} > 0$
 $\bar{r}_1 > 0 < > 0$

Center Manifold Theory (CMT)

$$\dot{x} = f(x)$$

$$x(t) \in \mathbb{R}^n$$

with Eq. point $\bar{x} = 0$

$$f(0) = 0$$

If $\dot{x} \neq 0$; i.e. $f(\bar{x}) = 0$ for $\bar{x} \neq 0$

$$z(t) := x(t) - \bar{x} \Rightarrow \dot{z} = \dot{x} - \dot{\bar{x}} = \dot{x} \Rightarrow$$

$$\dot{z} = f(x) = f(z + \bar{x})$$

$$\text{Since } f(\bar{x}) = 0 \Rightarrow \dot{z} = 0$$

Jacobians

$$\left. \frac{\partial f}{\partial x} \right|_{x=0}$$

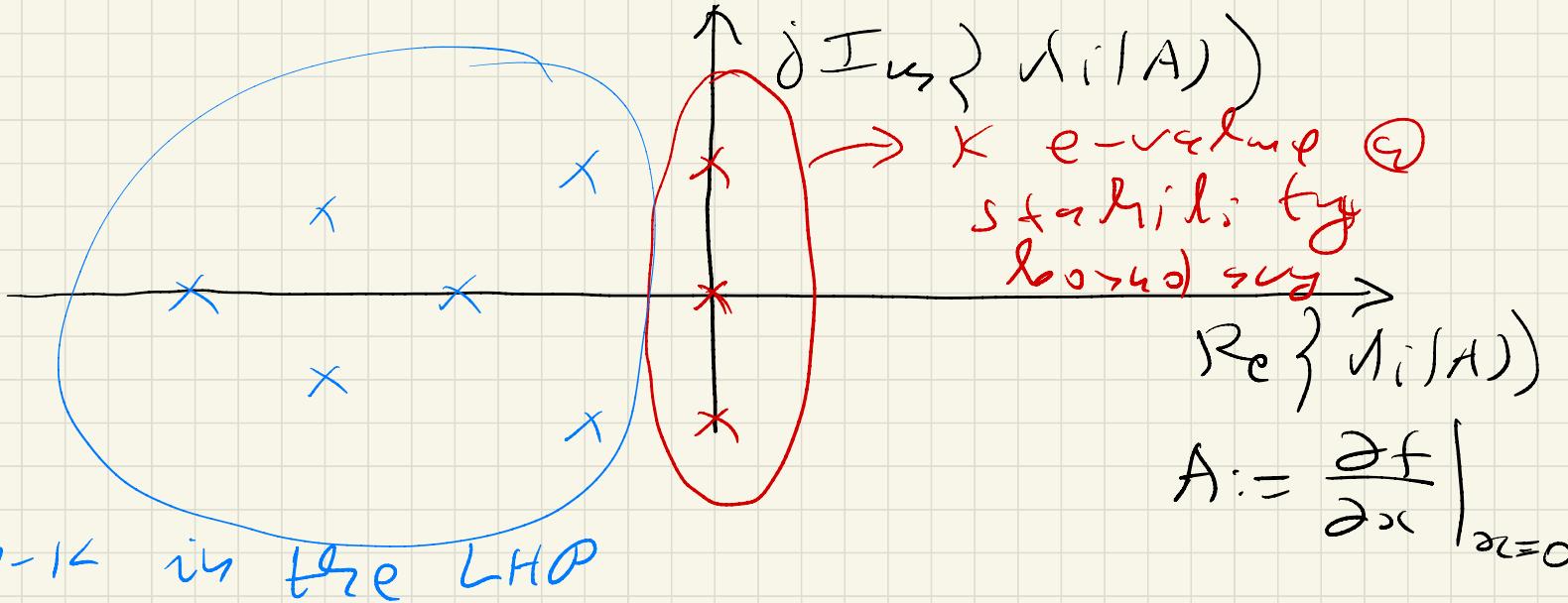
Let

A

have eigenvalues on $j\omega$ -axis

(with $n-k$ eigenvalues in LHP)

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Key: Linearization CANNOT be used to derive conclusions about stability properties of $\dot{x} = 0$ for $x = f(x)$.

IF $\dot{x} = f(x)^3$; $\dot{x} = 0 \cdot \tilde{x}$ $k=1$ $n-k=0$ II (5)
 \Rightarrow unstable
 \Rightarrow stable

CMT allows us to derive conclusions about stability of $\dot{x} = 0$ for this class of systems by examining stability of a system on a manifold of dimension

K (# of eigenvalues on jw-axis)

Rewrite $\dot{x} = f(x)$ as:

$$\dot{x} = \underbrace{Ax}_{\text{linearized part}} + \underbrace{\tilde{f}(x)}_{\text{Higher order non-linear terms}}$$

linearized part
of the dynamics

$$A = \frac{\partial f}{\partial x} \Big|_{x=0}$$

$$f(x) = f(0) + \frac{\partial f}{\partial x} \Big|_{x=0} x$$

Higher order
non-linear terms

$$\tilde{f}(x) = f(x) - Ax$$

$$f(x) = f(0) + \frac{\partial f}{\partial x} \Big|_{x=0} x + \text{H.O.T.}$$

⑥

Properties of $\tilde{f}(x) := f(x) - \frac{\partial f}{\partial x} \Big|_{x=0} \cdot x$

$$1^{\circ} \quad \tilde{f}(0) = 0$$

$$\tilde{f}(0) = f(0) - \frac{\partial f}{\partial x} \Big|_{x=0} \cdot 0 = 0$$

$$2^{\circ} \quad \left. \frac{\partial \tilde{f}}{\partial x} \right|_{x=0} = 0$$

$$\begin{aligned} & \left. \frac{\partial \tilde{f}}{\partial x} \right|_{x=0} = \left. \frac{\partial f}{\partial x} \right|_{x=0} - \left. \frac{\partial f}{\partial x_1} \right|_{x=0} \cdot \left. \frac{\partial x}{\partial x_1} \right|_{x=0} \\ & = 0 \end{aligned}$$

We are introducing a coordinate system that brings system \tilde{f} first to a form that is easier for analysis.

$$\dot{x} = Ax + \tilde{f}(x) ; \quad \tilde{f}(0) = 0 ; \quad \left. \frac{\partial \tilde{f}}{\partial x} \right|_{x=0} = 0$$

A: k e-values on jw-axis
 $\lambda_k - 0$ is in the LHP

Change of variables

$$x = \begin{bmatrix} y \\ z \end{bmatrix} ; \quad y(t) \in \mathbb{R}^k \\ z(t) \in \mathbb{R}^{n-k}$$

Note: We can choose τ to write A in $\begin{bmatrix} y \\ z \end{bmatrix}$ -coordinates as:

$$\bar{A} = \tau^{-1} A \tau = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right]$$

$$A_1 \in \mathbb{R}^{k \times k} : \operatorname{Re}\{\lambda_i(A_1)\} = 0 ; i=1, \dots, k$$

$$A_2 \in \mathbb{R}^{(n-k) \times (n-k)} : \operatorname{Re}\{\lambda_i(A_2)\} < 0 ; i=1, \dots, n-k$$

Can write \oplus as:

$$\begin{aligned} \dot{y} &= A_1 y + g_1(y, z) \\ \dot{z} &= A_2 z + g_2(y, z) \end{aligned} \quad \text{No coupling between } y \text{ & } z @ \text{ linear level}$$

From properties 1^o & 2^o of \tilde{f} \Rightarrow

$$g_i(0, 0) = 0$$

$i=1, 2$

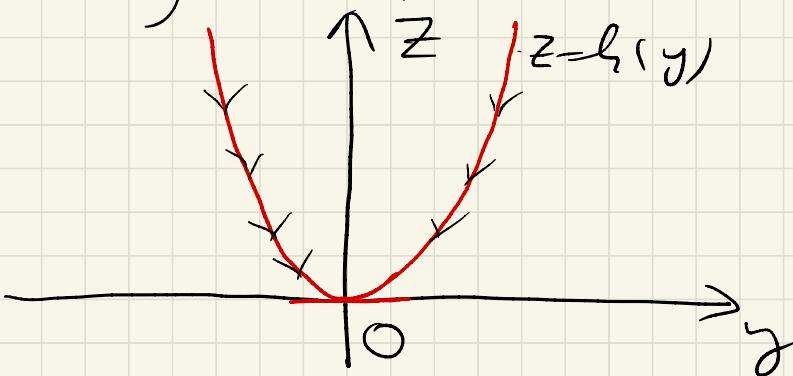
$$\frac{\partial g_i}{\partial y} \Big|_{(0, 0)} = 0$$

$$\frac{\partial g_i}{\partial z} \Big|_{(0, 0)} = 0$$

I_{Inv}: There is an invariant manifold
 $z = h(y)$ defined in the neighborhood of
 the origin (zero eq. point) such that

$$a) \quad h(0) = 0$$

$$b) \quad \frac{\partial h}{\partial y} \Big|_{y=0} = 0. \blacksquare$$



Invariant Manifolds:

$$W(t) := z(t) - h(y(t))$$

If $W(0) = 0 \Rightarrow W(t) = 0$ for all $t \geq 0$

If we start @ $z = h(y)$, we'll stay
 there forever.

