


Lecture 12

02/27/23

Today:

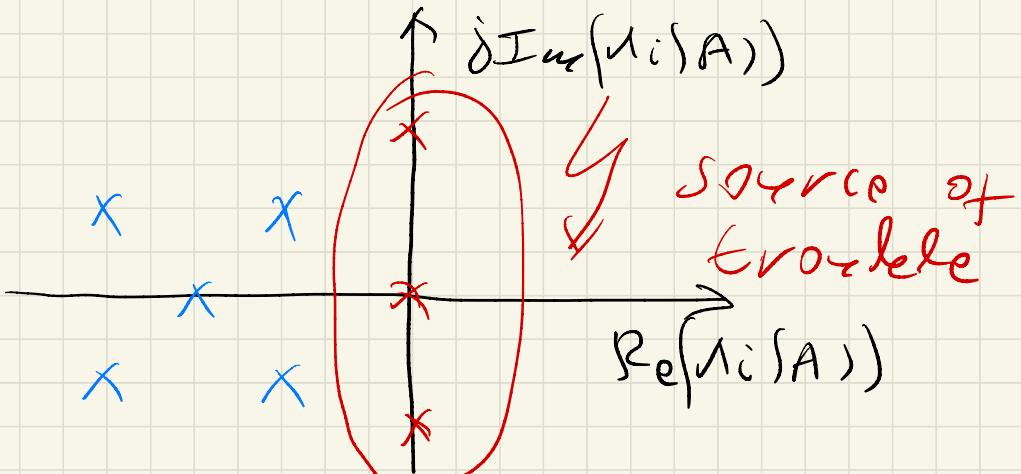
- Center Manifold Theory (continued)
- Existence & Uniqueness of solutions to
 $\dot{x} = f(x, t)$

→ A tool (method, technique) for studying stability properties of $\bar{x} = 0$ for $\dot{x} = f(\bar{x})$ [with $f(\bar{x}) = 0$] where Jacobians of linearizations around $\bar{x} = 0$

$$\left(A = \frac{\partial f}{\partial x} \Big|_{x=0} \right) \text{ has } k \text{ e-values on jw-axis}$$

$x(t) \in \mathbb{R}^n$

$m - k$ e-values in LHP ①



$$A = \frac{\partial f}{\partial x} \Bigg|_{x=0}$$

Linearization not conclusive: need to examine "strength" of nonlinear terms
(role)

Last time: coordinate transformations

$$x(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \rightarrow y(t) \in \mathbb{R}^k$$

$$x(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \rightarrow z(t) \in \mathbb{R}^{n-k}$$

leaving our system $\dot{x} = f(x)$ to:

②

$$\begin{cases} \dot{y} = A_1 y + g_1(y, z) \\ \dot{z} = A_2 z + g_2(y, z) \end{cases} \quad y(t) \in \mathbb{R}^k; \quad z(t) \in \mathbb{R}^{n-k}$$

$\operatorname{Re}(A_i) = 0; \quad i=1, \dots, k$

$\operatorname{Re}(A_2) < 0; \quad j=1, \dots, n-k$

"Simplifying" would mean terms:

$$g_i(0, 0) = 0; \quad \frac{\partial g_i}{\partial y} \Big|_{(0, 0)} = 0; \quad \frac{\partial g_i}{\partial z} \Big|_{(0, 0)} = 0$$

$$i=1, 2$$

Thus: There is an invariant manifold $z = h(y)$ in the neighborhood of the zero eq. point s.t.

a) $h(0) = 0$; b) $\frac{\partial h}{\partial y} \Big|_{y=0} = 0$

(3)

Invariance: $z(0) = h(y(0)) \Rightarrow$
 $z(t) = h(y(t))$ for all $t \geq 0$

Thm: If $\bar{y} = 0$ is an asymptotically stable (unstable) eq. point of

$$\dot{y} = A_1 y + [g_1(y, h(y))]$$
: k -dimensional subsystem

then $\bar{x} = 0$ is an asymptotically stable (unstable) eq. point of

$$\dot{x} = f(x); \quad x(t) \in \mathbb{R}^n$$

$$y(t) \in \mathbb{R}^k$$

 $k < n$

Remaining tasks:

- Characterization of invariant manifold

New variable: $\left\{ \begin{array}{l} z = h(y) \\ w(t) := z(t) - h(y(t)) \end{array} \right.$

$w(0) = 0 \Rightarrow w(t) = 0 \text{ for all } t \geq 0$

i.e. $w(t) \equiv 0 \Rightarrow \frac{dw(t)}{dt} \equiv 0$

$$\dot{w} = \dot{z} - \frac{\partial h}{\partial y} \cdot \dot{y} \equiv 0 \rightarrow \text{on the invariant manifold } (z = h(y))$$

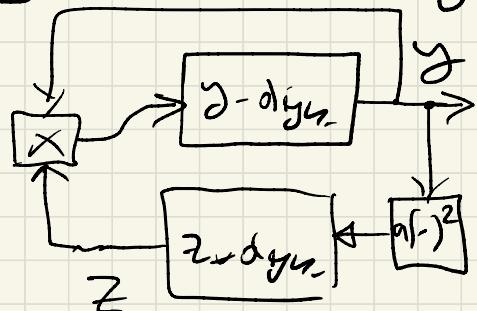
$$0 = A_2 \cdot h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y} [A_1 y + g_1(y, h(y))]$$

with: $h(0) = 0 ; \frac{\partial h}{\partial y} |_{y=0} = 0$

Challenge: We have to solve this eq. to find $h(y)$ [HARD]

Ex: $n=2$; $k=1$; $\gamma(t) \in \mathbb{R}$; $z(t) \in \mathbb{R}$; σ : constant parameter

$$\begin{aligned} \dot{y} &= \sigma \cdot y + y \cdot z \\ \dot{z} &= -z + \sigma \cdot y^2 \end{aligned} \quad \left. \begin{array}{l} A_1 = 0; A_2 = -1 \\ g_1(y, z) = y \cdot z; g_2(y, z) = \sigma \cdot y^2 \end{array} \right\}$$



$$\begin{aligned} 0 &= -h(y) + \sigma \cdot y^2 - \frac{\partial h}{\partial y} [y \cdot h(y)] \\ y \cdot h(y) \cdot \frac{dh}{dy} &= -h(y) + \sigma \cdot y^2 \\ h(0) = 0; \frac{dh}{dy} \Big|_{y=0} &= 0 \end{aligned}$$

Difficult to solve for $h(y)!!$



$$y \cdot h(y) \frac{\partial h}{\partial y} = -h(y) + ay^2; \quad h(0) = \left. \frac{\partial h}{\partial y} \right|_{y=0} = 0$$

Approach: Use Taylor series expansion of $h(y)$ around $\bar{y} = 0$:

$$h(y) = h(0) + \frac{\partial h}{\partial y} \Big|_0 \cdot y + \frac{1}{2} \frac{\partial^2 h}{\partial y^2} \Big|_0 \cdot y^2 + \\ + \frac{1}{3!} \frac{\partial^3 h}{\partial y^3} \Big|_0 \cdot y^3 + \dots$$

$$h(y) = \sum_{k=2}^{\infty} h_k \cdot y^k = h_2 \cdot y^2 + h_3 \cdot y^3 + \dots$$

$$\frac{\partial h(y)}{\partial y} = \sum_{k=2}^{\infty} k h_k y^{k-1} = 2h_2 y + 3h_3 y^2 + \dots$$

Substitute h and $\frac{dh}{dy}$ into $\textcircled{1}$ and evaluate
equal powers in y :

$$y \left[h_2 y^2 + h_3 y^3 + \dots \right] \left[2 \cdot h_2 y + 3 \cdot h_3 y^2 + \dots \right] =$$

$$= - \left[h_2 y^2 + h_3 y^3 + \dots \right] + \boxed{a \cdot y^2}$$

$$\mathcal{O}(y^2): h_2 = a$$

$$\mathcal{O}(y^3): h_3 = 0$$

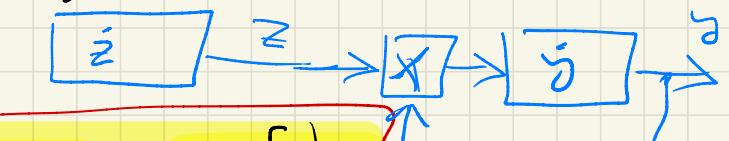
$$\mathcal{O}(y^4): h_4 = -2h_2^2 = -2a^2$$

$$h(y) = a \cdot y^2 - 2a^2 y^4 + \mathcal{O}(y^5)$$

Substitution of these expressions into

$$\dot{y} = y \cdot h(y) \text{ gives:}$$

$$\dot{y} = a \cdot y^3 - 2a^2 y^5 + \mathcal{O}(y^6)$$



(8)

Local stability properties of $\bar{y} = 0$
determined by $\dot{y} = \boxed{a} y^3$

$a < 0 \Rightarrow \bar{y} = 0$ is Locally Asymptotically Stable (LAS)

$a > 0 \Rightarrow \bar{y} = 0$ is UNSTABLE

IF Note : For $a = 0$:

$$\dot{y} = y \cdot z$$

$$\dot{z} = -z + 0 \cdot y^2 \Rightarrow \dot{z} = -z \Rightarrow z(t) = z(0) e^{-t}$$

$$\dot{y} = z(0) \cdot e^{-t} \cdot y$$

Time-varying
Linear Systems

Existence & Uniqueness of solutions

$\dot{x} = f(x, t)$: time-varying coordinate system

If linear case: $\dot{x} = \underbrace{A(t)}_{[a_{ij}(t)]_{n \times n}} x$

Q: What kind of functions $a_{ij}(t)$ would allow us to conclude existence & uniqueness of solutions?

If $A(t) = A = \text{const. } n \times n \text{ matrix}$

$$x(t) = e^{At} \cdot x(0) = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} x(0)$$

unique solution