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## Lecture 13

03/01/23

- Existence & Uniqueness of solutions  
to  $\dot{x} = f(x, t)$  (Well-posedness)

If Linear algebra (static case)

$$A \dot{x} = b \dots (\text{LA}) ; x \in \mathbb{R}^n ; A \in \mathbb{R}^{m \times n}$$

- a) Existence: there is  $\Leftrightarrow$  solution to (LA)

$$\begin{bmatrix} a_1 & a_2 & \dots & a_m \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \sum_{i=1}^m x_i \begin{bmatrix} a_i \end{bmatrix}$$

If  $b \in \text{Range}(A)$   $\Rightarrow$  column span of  $A \Rightarrow$   
 $\Rightarrow$  there is  $\Leftrightarrow$   $x$  that satisfies (LA)

- b) Uniqueness:  $\text{Null}(A) = \{0\} \in \mathbb{R}^n$

If  $\text{Null}(A) \neq \{0\} \Rightarrow$  any  $\tilde{x} \in \text{Null}(A)$   
 can be used to obtain another solution  
 i.e. if  $A\tilde{x} = b \Rightarrow A(\tilde{x} + \tilde{\alpha}) = b$

$\tilde{\alpha}$  is a solution to  $(A)$   $\Rightarrow$  so is  $\tilde{x} + \tilde{\alpha}$

Fact (EE 535):  $\dot{x} = A(t)x$ ;  $f(x, t) = A(t)x$

If each element of the matrix  $A(t)$  is a  
 $a_{ij}(t)$

piecewise cont function of time  $\Rightarrow$

$\dot{x} = A(t)x$  has a unique solution

[Starting at  $x(t_0) = x_0$  for any  $t \geq t_0$ )

$$x(t) = \Phi(t, t_0)x(t_0);$$



$$\frac{\partial \Phi(t, t_0)}{\partial t} = A(t)\Phi(t, t_0)$$

$$\Phi(t_0, t_0) = I$$

②

Goal: Identify the class of functions  $f(x, t)$   
for which there is a solution to

$$\dot{x} = f(x, t) \dots \text{at}$$

and this solution is unique on some time  
interval  $[t_0, t_f]$

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~~(1)~~: 1st order (in time) nonlinear ODE

$t$ : time ( $t \geq t_0$ )

$x$ : state [ $x(t) \in \mathbb{R}^n$ ]

$f(x, t)$ : nonlinear function of  $x$  &  $t$

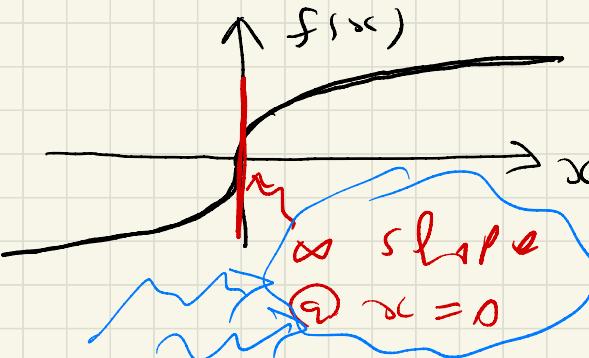
$$f: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$$

We'll assume that  $f$  is a piecewise cts  
function of time [as in the linear case]

Q: Is it enough to consider class of  
functions  $f$  that are cts w.r.t.  $\dot{x}$ ?

Fact 1: A: Yes, for existence  
No, for uniqueness

$$\text{Ex: } \dot{x} = x^{\frac{1}{3}} = \sqrt[3]{x}; \quad x(t) \in \mathbb{R}$$



$$x(0) = 0 \Rightarrow x(t) = 0 \text{ is a solution}$$

BUT, there is another solution that starts  
at  $x(0) = 0$ !!!

$$x(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}} \Rightarrow x(0) = 0$$

$$\dot{x} = \frac{3}{2} \left(\frac{2t}{3}\right)^{\frac{3}{2}-1} \cdot \frac{2}{3} = \left(\frac{2t}{3}\right)^{\frac{1}{2}} = x^{\frac{1}{3}} \Rightarrow$$

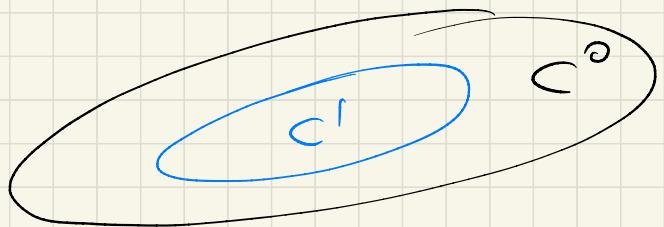
$x(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}}$  satisfies  $\dot{x} = x^{\frac{1}{3}}$  with  $x(0) = 0$

So does  $x(t) = 0$ !!!

Note:  $f(x) = x^{\frac{1}{3}}$  is NOT cts & diffble  
 (continuously differentiable)

$$\frac{df}{dx} = \frac{1}{3} \cdot x^{-\frac{2}{3}} = \frac{1}{3 \sqrt[3]{x^2}} \xrightarrow{x \rightarrow 0} +\infty$$





- $C^0$ : a class of cts function
- $C^1$ : a class of diffble functions with cts first derivative  
(cts diffble)

## Lipschitz continuity

A function  $f$  is  $L$ -cts if

$$|f(x) - f(y)| \leq L \cdot |x - y|$$

$\frac{L}{2}$

Lipschitz constant

Global  $L$ -cts: If there is  $L > 0$  s.t.  
 $(Lc)$  holds for all  $x, y \in \mathbb{R}^n$   
[too strong]

$$\text{Ex: } f(x) = x^2$$

$$|x^2 - y^2| = |(x-y)(x+y)| \leq \underbrace{|x+y|}_{L(x,y)} \cdot |x-y|$$

There is NO uniform bound on  $L(x,y)$   
(i.e. there is NO  $L > 0$  s.t.  $(L)$  holds  
for all  $x, y \in \mathbb{R}$ )

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$$\text{Ex: } f(x) = x^3$$

$$|x^3 - y^3| \leq |x^2 + xy + y^2| \cdot |x-y|$$

NOT globally L-Lips.

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Good news: BOTH functions are  
LOCALLY L-Lips (All we need)  $\Theta$

Local L-cts: If there is  $L > 0$  s.t.  
 $|Lc|$  holds for all  $x, y \in B_\delta(\bar{x})$

$$B_\delta(\bar{x}) := \{ z \in \mathbb{R}^n \text{ s.t. } \|z - \bar{x}\| \leq \delta \}$$

Note: Both  $f(x) = x^2$  &  $f(x) = x^3$  are  
Locally L-cts!!!

Fact 2: If  $f(x, t)$  is Locally L-cts w.r.t  $x$   
then we have existence & uniqueness of  
solutions on a finite time interval  $[t_0, t_f]$

Fact 3: ————— || ————— GLOBALLY L-cts w.r.t.  $x$   
———— | | ————— ) ) —————  
———— | | ————— ) ) —————  
on  $[t_0, \infty)$  (3)

Facts 2 & 3: Theorems that provide sufficient conditions for existence & uniqueness of solutions to  $\dot{x} = f(x, t)$

IF Along with piecewise continuity of  $f$  w.r.t time  $t$ ,

Fact 4: Any cts diffble function  $f(x)$  is locally L-cts.

$$\text{Ex: } f(x) = x^p ; p \geq 1 \quad \frac{\partial f}{\partial x} = p \cdot x^{p-1} \quad \text{Locally L-cts}$$

$$\text{Ex: } f(x) = \sin(x) \Rightarrow \frac{\partial f}{\partial x} = \cos(x)$$

globally L-cts

$$\text{Ex: } f(x) = x^p ; p \in (0, 1) \quad @ x = 0$$

Not cts diffble

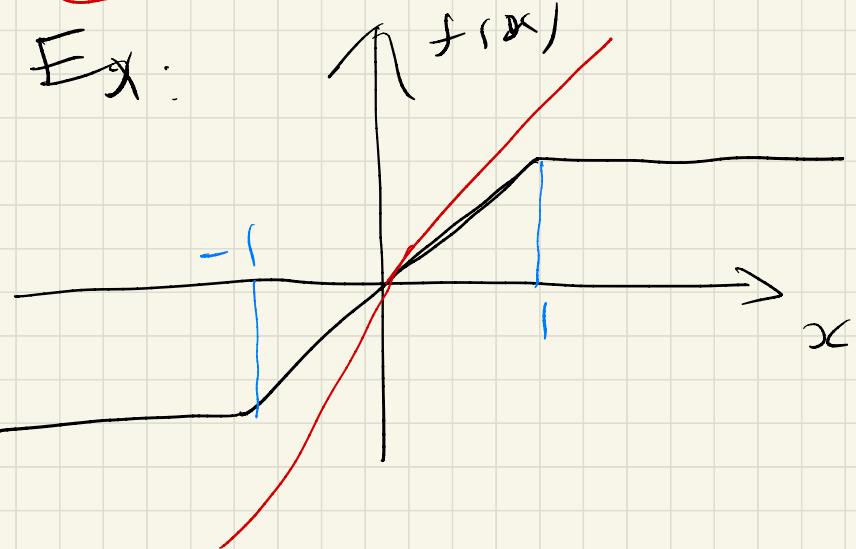
(G)

For a function to be globally Lcts

1 sufficient condition is uniform boundedness of  $\left\| \frac{\partial f}{\partial x} \right\|$ .

If there is  $L > 0$  s.t.  $\left\| \frac{\partial f}{\partial x} \right\| < L$  for all  $x \in \mathbb{R}$   $\Rightarrow$   $f$  is GLOBALLY Lcts.

Ex:

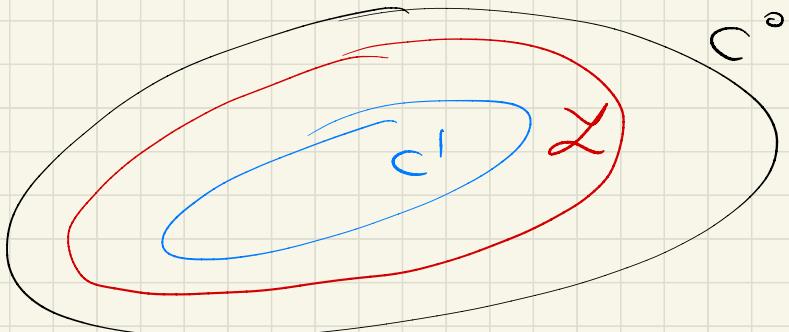


GLOBALLY Lcts

but not  
differentiable @

$$x = \pm 1$$





$Z - L - \text{cts}$  functions

$$C^1 \subset Z \subset C^0$$

cts

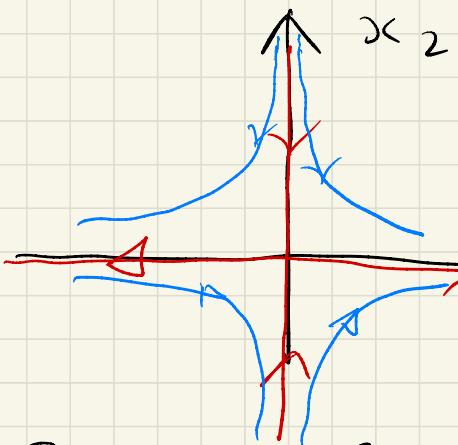
cts diffble

functions

Continuous dependence of I.C.'s & parameters

Assume existence & uniqueness of  
solutions to  $\dot{x} = f(x)$ ;  $x(0) = x_0$   
on  $[0, t_f]$  for some  $t_f$ .

Q: Can we guarantee continuity  
w.r.t. to I.C.s on some time  
interval?



Even in linear case,  
to much to expect  
continuity w.r.t I.C.  
for  $[0, \infty)$

Given  $\xi > 0$ ,  $\exists \delta > 0$  s.t.

$$\forall x_0 \in \{x \in \mathbb{R}^n \text{ s.t. } \|x - x_0\| < \delta\} = B_\delta(x_0)$$

1°  $\phi(x_0, t)$  is a unique solution,

$$2° \|\phi(x_0, t) - \phi(x_0, t)\| < \xi$$

$x(t) \rightarrow$  all  $t \in [t_0, t_f]$   
starting @  $x(0) = x_0$

(12)

Good news: continuity w.r.t. ICs on  
some finite time interval (comes  
for free [from existence & uniqueness  
of solutions])