

---

---

---

---

---



# Lecture 14

03/06/23

Last time:

Existence & Uniqueness  
of solutions to  
 $\dot{x} = f(x, t)$

$\Rightarrow$

Contingency  
u.v.t. I.C.'s  
and  
parameters

→ today

$$\begin{cases} \dot{x} = f(x, u, t) \end{cases}$$

$u \in \mathbb{R}^p$ : vector of constant parameters

→ Rewrite as:  $z := \begin{bmatrix} x \\ u \end{bmatrix}$

$$\begin{cases} \dot{x} = f(x, u, t) \\ \dot{u} = 0 \end{cases} \quad \dot{z} = F(z, t) = \begin{bmatrix} f(z, t) \\ 0 \end{bmatrix}$$

If  $f$  is locally L-cts w.r.t  $\bar{z} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow$   
continuity w.r.t. parameters [because of  
— II — w.r.t. I.C.]

## Today: Sensitivity Equations

(a tool for examining sensitivity of  
solutions of  $\dot{x} = f(x, \bar{y}, t)$  to changes in  
parameters  $\bar{y}$ )

$$\dot{\bar{x}} = f(\bar{x}, \bar{y}, t) \quad \textcircled{*} \quad \bar{x}(t) \in \mathbb{R}^n \text{ state vector}$$

$\bar{y} \in \mathbb{R}^p$  vector of

const. parameters

Setup: For  $\bar{y}$  the solution to  $\textcircled{*}$   
from  $\bar{x}(t_0) = \bar{x}_0$  is given by  $\bar{x}(\bar{y}, t)$   $\textcircled{2}$

Q: Determine impact of change in  $M$   
to  $\alpha(\bar{M}, t)$

$$M = \bar{M} \rightarrow \alpha(\bar{M}, t) \quad \text{given}$$

$$M = \bar{M} + \tilde{M} \rightarrow \alpha(\bar{M} + \tilde{M}, t) = ?$$

$$\alpha(\bar{M} + \tilde{M}, t) = \alpha(\bar{M}, t) + \left[ \frac{\partial \alpha(\bar{M}, t)}{\partial M} \right] \cdot \tilde{M} +$$

→ Taylor Series

expression of

$\alpha(M, t)$  around  $\bar{M}$

$$+ O\left(\|\tilde{M}\|^2\right)$$

H.O.T.

$$x(\sqrt{m} + \tilde{\mu}, t) \approx x(\sqrt{m}, t) + S(t) \cdot \tilde{\mu}$$

H.O.T. terms been neglected

$$S(t) := \left. \frac{\partial x(m, t)}{\partial m} \right|_{m=\bar{m}} = x_m(\bar{m}, t) \Big|_{m=\bar{m}}$$

Sensitivity function

(matrix valued)

$$\text{Ex: } x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; \quad \bar{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

$$S(t) = \left. \begin{bmatrix} \frac{\partial x_1}{\partial m_1} & \frac{\partial x_1}{\partial m_2} & \frac{\partial x_1}{\partial m_3} \\ \frac{\partial x_2}{\partial m_1} & \frac{\partial x_2}{\partial m_2} & \frac{\partial x_2}{\partial m_3} \end{bmatrix} \right|_{m=\bar{m}}$$

(4)

Objective: Find equations that governs the evolution of  $\dot{x}(t)$  for  $\dot{x} = f(x, m, t)$

$$x(m, t) = x_0 + \int_{t_0}^t f(x, m, \tau) d\tau$$

$$x(t_0)$$

differentiate

first w.r.t.  $m$

then w.r.t.  $t$

$$\frac{\partial x(m, t)}{\partial m} \Big|_{m=\bar{m}} = \frac{\partial x_0}{\partial m} + \int_{t_0}^t \left[ \frac{\partial f}{\partial x} \frac{\partial x}{\partial m} + \frac{\partial f}{\partial m} \right] \cdot d\tau$$

$$S(t)$$

$$A(\tau)$$

$$B(\tau)$$

$$m=\bar{m}$$

Thus:

$$S(t) = \int_0^t [A(\tau)S(\tau) + B(\tau)] d\tau$$

$$A(\tau) := \frac{\partial f}{\partial x} \Big|_{x(\bar{v}, \tau), \bar{v}} ; \quad B(\tau) = \frac{\partial f}{\partial v} \Big|_{x(\bar{v}, \tau), \bar{v}}$$

Sensitivity Equation

$$\Rightarrow \frac{d}{dt} S(t) = A(t)S(t) + B(t)$$

Matrix-valued linear time-varying ODE  
for  $S(t)$

Ex: Fold bifurcation

$$\dot{x} = x^2 + \sqrt{m} = f(x, m)$$

$$\frac{\partial f}{\partial x} = 2 \cdot x ; \quad \frac{\partial f}{\partial m} = 1 \Rightarrow \frac{dS(t)}{dt} = 2 \cdot x(\sqrt{m}, t) \cdot S(t) + 1$$

"Implementation":

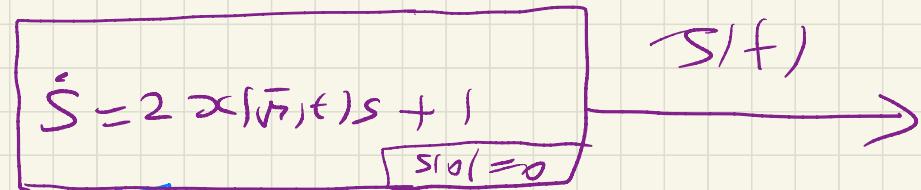
$$\begin{aligned}\dot{x} &= x^2 + \sqrt{m} \\ \dot{S} &= 2 \cdot x \cdot S + 1\end{aligned}$$

One way coupling  
(from  $x$  to  $S$ )

Block diagram:

$$\begin{cases} \dot{x} = x^2 + \sqrt{m} \\ x(0) = x_0 \end{cases}$$

$$x(\sqrt{m}, t)$$



$$x(\sqrt{m} + \tilde{m}, t) \approx x(\sqrt{m}, t) + S(t) \sqrt{m}$$

$$\sqrt{m} = \bar{m}$$

selection for

②

Ex : From Khalil

$$\dot{x}_1 = \dot{x}_2 = f_1(x_1, x_2, v)$$

$$\dot{x}_2 = -c \cdot \sin(x_1) - (a + b \cdot \cos(x_1)) \cdot x_2 = f_2(x_1, x_2, v)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2 ; \quad v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$$

$$f(t) \in \mathbb{R}^{2 \times 3} ; \quad S(t) : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

Study sensitivity for  $\bar{v} = [1 \ 0 \ 1]^T$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c \cdot \cos(x_1) + b \cdot x_2 \sin(x_1) & -[a + b \cdot \cos(x_1)] \end{bmatrix}$$

$$\left. \frac{\partial f}{\partial x} \right|_{\bar{v}} = \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & -1 \end{bmatrix}$$

$$\frac{\partial f}{\partial v_i} = \begin{bmatrix} \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial c} \\ \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial c} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2 \cos x_1 & -\sin x_1 \end{bmatrix}$$

$$S(t) = \begin{bmatrix} S_a(t) & S_b(t) & S_c(t) \end{bmatrix} =$$

$$= \begin{bmatrix} s_{1a}(t) & s_{1b}(t) & s_{1c}(t) \\ s_{2a}(t) & s_{2b}(t) & s_{2c}(t) \end{bmatrix} + \frac{\partial f}{\partial v_i} \tilde{v}_i$$

Need to form  $A(t) \cdot S(t) + B(t) = \frac{\partial f}{\partial v_i} \int_M f^i(t) + g$

$s_{i|t}$ : sensitivity of  $x_i(\sqrt{n}, t)$  to a  
 $i=1, 2$

$$A(t) \cdot s_i(t) = c_i(t)$$

$$\dot{s}_i(t) = c_i(t) + \beta_i(t)$$

$$\begin{bmatrix} \dot{s}_a(t) & \dot{s}_g(t) & \dot{s}_c(t) \end{bmatrix} = \begin{bmatrix} c_a(t) + \beta_a(t) \\ c_g(t) + \beta_g(t) \\ c_c(t) + \beta_c(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{s}_a \\ \dot{s}_g \\ \dot{s}_c \end{bmatrix}_{\text{ex}} = \begin{bmatrix} \frac{c_a + \beta_a}{c_g + \beta_g} \\ \frac{c_g + \beta_g}{c_c + \beta_c} \\ \frac{c_c + \beta_c}{c_g + \beta_g} \end{bmatrix}_{\text{ex}}$$

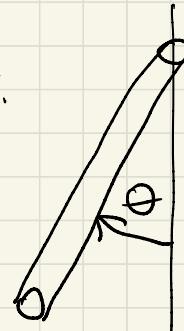
(1D)

Stability: Many different notions

st eq. points of  $\dot{x} = f(x)$ ; H. is not  
(w.r.t. ICS)

→ (in the sense of Lyapunov)

Ex:



Two Eq. points:

$$\text{down: } \begin{bmatrix} \bar{x} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{up: } \begin{bmatrix} \bar{x} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

Q: How trajectories around eq. points  
evolve as a function of time

QUALITATIVELY

w/o solving equations

$\dot{x} = f(x)$  : Locally L-cts &  
Eq. point @  $\bar{x} = 0$

Def:  $\bar{x} = 0$  of  $\dot{x} = f(x)$  is

1° STABLE (in the sense of Lyapunov) if

$\forall \varepsilon > 0$ ,  $\exists \delta, (\varepsilon) > 0$  [ $\delta, (\varepsilon) < \varepsilon$ ] s.t.

for every  $x_0$  there is

$$\|x(t_0)\| < \delta, (\varepsilon) \Rightarrow \|x(t, x_0)\| < \varepsilon$$

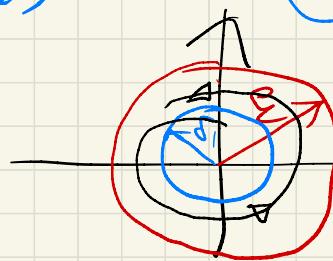
$x_0$

$$\|x(t_0) - \bar{x}\| < \delta, (\varepsilon)$$

start close

stay close

(to  $\bar{x}$ )



for all  $t \geq 0$ .

$$\|x(t, x_0) - \bar{x}\| < \varepsilon$$

No requirement +  
attract convergence  
 $\Leftrightarrow \bar{x} = 0$  (12)

2° UNSTABLE : if  $\lambda^0$  does NOT hold

3° LOCALLY ASYMPTOTICALLY STABLE

if  $\lambda^0$  holds and (LAS)

$$\exists \delta_2 > 0 \text{ s.t. } \|x_0\| < \delta_2 \Rightarrow \|x_0 - \bar{x}\| \lim_{t \rightarrow \infty} \|x(t, x_0)\| = 0$$
$$\|x(t, x_0) - \bar{x}\|$$

LAS = STABLE  $\oplus$  ATTRACTIVE

4° GLOBALLY ASYMPTOTICALLY STABLE (GAS)

If 3° holds for  $\delta_2 = +\infty$ . (Q3)