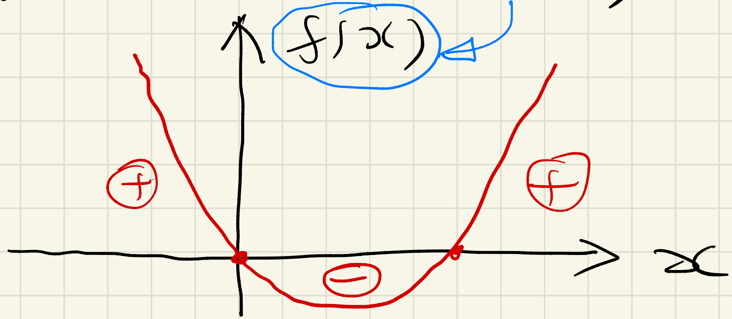
Lecture 15

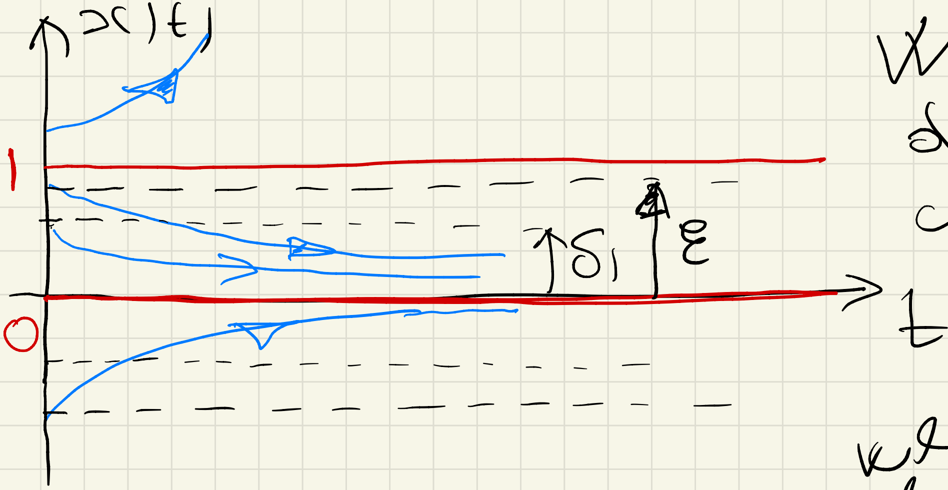
03/08/23

Today:

- Use of stability definitions (examples)
- Lyapunov-based method for stability analysis

Ex: $\dot{x} = \underbrace{x(x-1)}_{f(x)}$ $\Rightarrow$ 2 eq. points $\bar{x} = 0$; $\bar{x} = 1$





We can use stability definition to directly check stability properties of eq. points in situations when we can determine solution to $\dot{x} = f(x)$

In this example, $\bar{x} = 1$ is unstable

$\bar{x} = 0$: LAS (locally asymptotically stable)

Ex: Harmonic oscillator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; \quad \ddot{y} + y = 0$$

$x_1 = y ; x_2 = \dot{y}$

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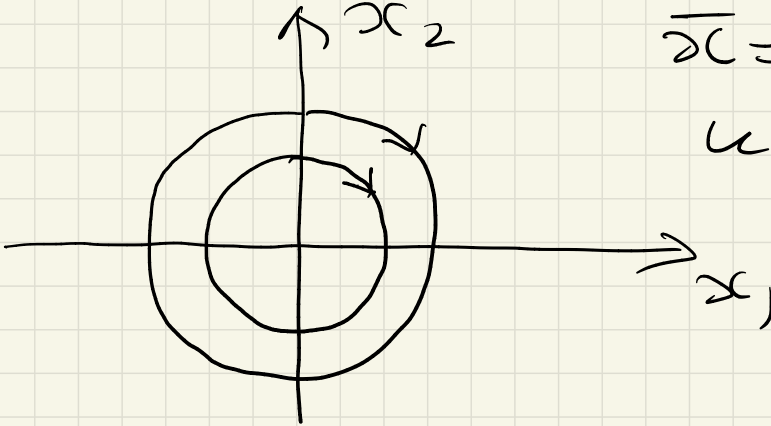


$\bar{x}=0$; stable but not asymptotically stable.

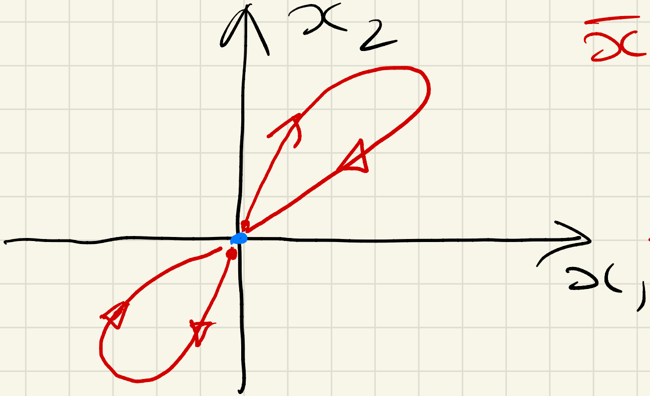
$\cancel{\left(x(t) \xrightarrow{t\to\infty} 0 \right)}$

For linear systems,

$$\lim_{t\to\infty} \|x(t)\| = 0 \Rightarrow \bar{x}=0 \text{ is stable eq. point.}$$

In nonlinear case, attractiveness of $\bar{x}=0$ (i.e., $\lim_{t\to\infty} \|x(t)\| = 0$) DOES NOT imply stability of $\bar{x}=0$.

Ex: Homoclinic orbit



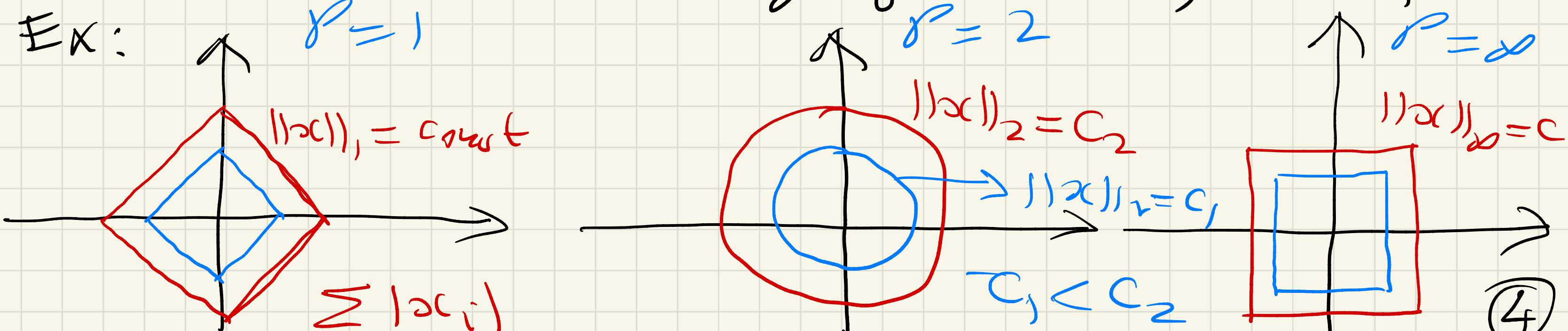
$\bar{x} = 0$ is not stable but

$$\lim_{t \to \infty} \|x(t)\| = 0$$

See Khalil for details

Note on norms: $\|\cdot\|$ in stability definition is any p-norm; $p \geq 1$

Ex:



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Lyapunov-based method for stability of eq. points of $\dot{x} = f(x)$

Motivating example (pendulum)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -a \sin x_1 - b x_2 \end{bmatrix} \qquad x_1 = \theta, \quad x_2 = \dot{\theta}$$

$$\ddot{\theta} + b \cdot \dot{\theta} + a \cdot \sin\theta = 0$$

Eq. points: $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$

"down" "up"

Neither eq. point CAN be GAS (because of the existence of another eq. point!!!)

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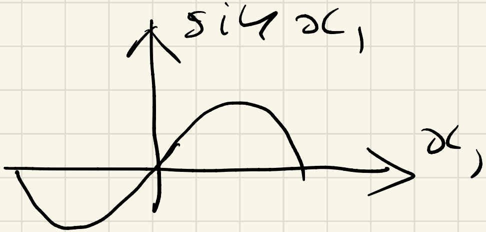
Energy:

$$E(x_1,x_2) = \underbrace{c_1 \cdot \int_0^{x_1} \sin\xi \, d\xi}_{\text{potential}} + \underbrace{c_2 \cdot \frac{1}{2} x_2^2}_{\text{kinetic}}$$

$c_1(1-\cos x_1)$

$\to E(0,0) = 0$

$E(x_1,x_2) > 0$ on domain D around the origin



$x_1 \sin x_1 > 0$, $x_1 \in (-\pi, \pi)$

Objective: study $\dfrac{dE}{dt}$ along solutions to $\dot{x} = f(x)$.

$$E(t) = E(x_1(t), x_2(t))$$

$$\boxed{\frac{dE}{dt}} = [\nabla E]^T \cdot \frac{dx}{dt} = \underbrace{\begin{bmatrix} \frac{\partial E}{\partial x_1} & \frac{\partial E}{\partial x_2} \end{bmatrix}}_{[\nabla E]^T} \cdot \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial E}{\partial x_1} & \frac{\partial E}{\partial x_2} \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = f_1 \frac{\partial E}{\partial x_1} + f_2 \frac{\partial E}{\partial x_2}$$

$$\boxed{= \sum_{i=1}^{2} f_i(x_1, x_2) \cdot \frac{\partial E(x_1, x_2)}{\partial x_i}}$$

Holds for any 2nd order system

For pendulum:

$$\frac{\partial E}{\partial x_1} = c_1 \cdot \sin x_1 \; ; \; \frac{\partial E}{\partial x_2} = c_2 \cdot x_2$$

$$f_1 = x_2 \; ; \; f_2 = -a \sin x_1 - b x_2$$

$\Rightarrow$?

$$\frac{dE}{dt} = c_1 \cdot x_2 \cdot \sin x_1 - c_2 x_2 \{a \sin x_1 + b x_2\}$$

$$= \underbrace{[c_1 - c_2 \cdot a]}\, \cdot x_2 \sin x_1 - c_2 \cdot b \cdot x_2^2$$

sign-indefinite
(i.e. can be both positive
and negative around
$x = 0$)

(always
negative)

The best we can do: choose c_1 and c_2
to cancel it.

E.g. $c_2 = 1$; $c_1 = a$ works !!!

With this choice $\Rightarrow$ $\dfrac{dE}{dt} = -b \cdot x_2^2 \leq 0$

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Note: $\dfrac{dE}{dt} = 0 \cdot x_1^2 - b \cdot x_2^2 \leq 0$ on D

$x_1 \neq 0$ and $x_2 = 0 \Rightarrow \dfrac{dE}{dt} = 0$

$\Rightarrow$ $E(t)$ is a non-increasing function of time.

Allows us to conclude stability (in the sense of Lyapunov) of $\bar{x} = 0$.

(but not local asymptotic stability): further analysis or different choice of energy-like function required.

If $k=0$ $\Rightarrow$ no viscous damping

$$\frac{dE}{dt} \equiv 0 \Rightarrow E(t) = E(t_0) = \text{const}.$$

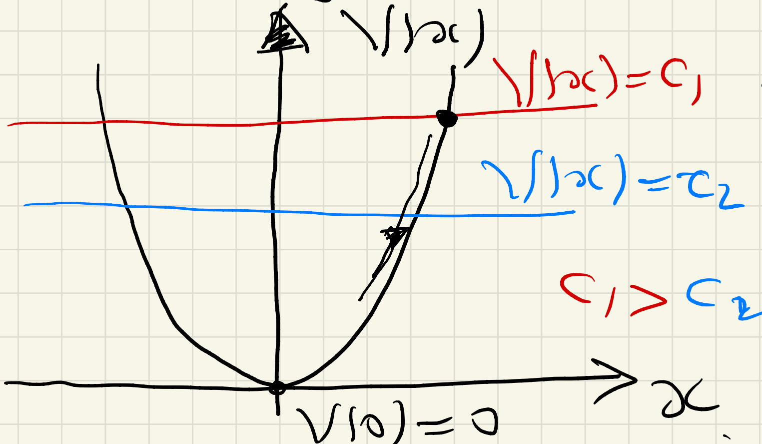
Conservative system [Energy does not increase or decrease along the solutions]

(Same as harmonic oscilator).

linearization of pendulum with $k=0$ around $\bar{x}=0$.

Lyapunov-based method for stability analysis of eq. points generalizes the above argument to a broader class of energy-like functions for $\dot{x}=f(x)$.

Ex: $\dot{x} = f(x)$ with $x(t) \in \mathbb{R}$



Lyapunov function candidate

$V(x)$ [instead of $E(x)$]

- represents an initial conditions

want to figure out if V increases; decreases or stays constant along the solutions of $\dot{x} = f(x)$

Ex: $\dot{x} = x(x-1)$

$V(x) = \frac{1}{2}x^2$ $\left(V(0) = 0 \; ; \; V(x) > 0 \text{ for all } x \neq 0 \right)$

$\dot{V} = \frac{\partial V}{\partial x} \cdot \dot{x} = x \cdot x(x-1) =$

$= -\underbrace{(1-x)}_{\text{want it to be } > 0 \text{ for } x \neq 0} x^2$

$f(x)$

For $x < 1 \Rightarrow \frac{dV}{dt} < 0$ for all $x \neq 0$

$\frac{dV}{dt} = 0$ for $x = 0$ [by construction 2

Thus:

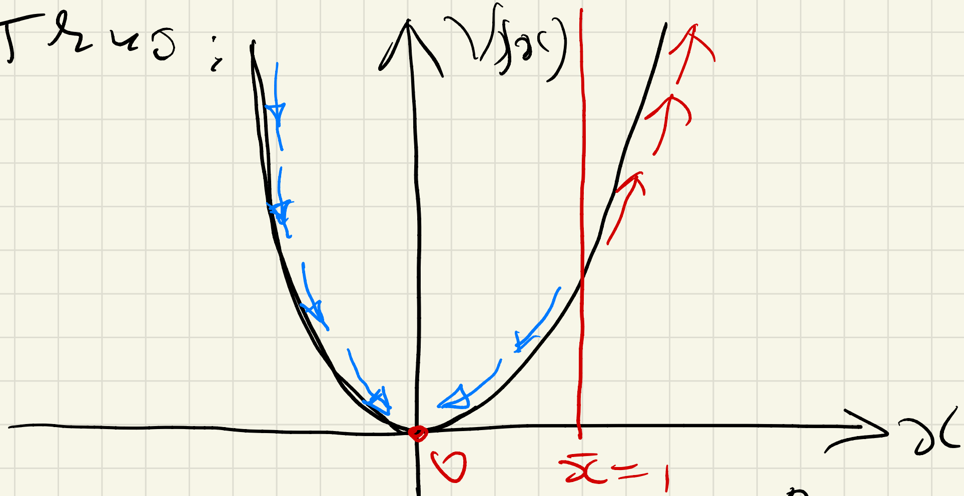
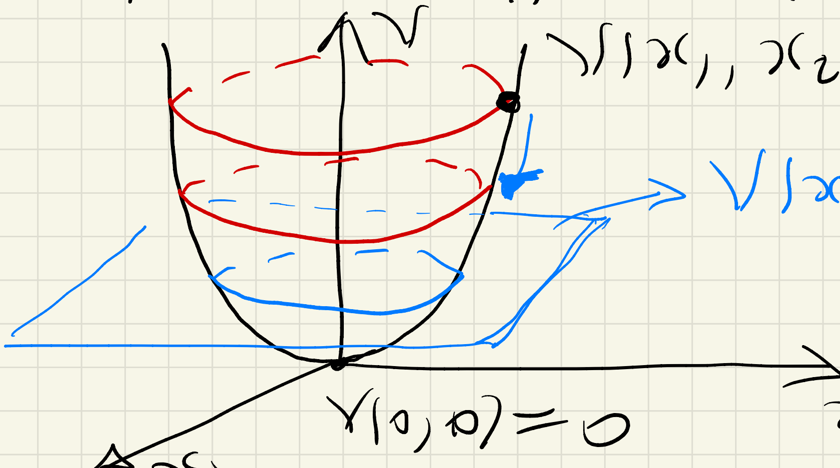


Illustration in $\mathbb{R}^2$ $\quad V(x_1, x_2)$

$V(x_1, x_2) > 0$ for $x \neq 0$



$V(x_1, x_2) = \text{const}$

Thm: Let $\mathcal{D}$ be an open connected subset of $\mathbb{R}^n$ that contains the eq. point $\bar{x}=0$ of $\dot{x}=f(x)$ $[f(0)=0]$.

1° If there is a continuously differentiable function

$$V:\mathcal{D}\longrightarrow\mathbb{R}$$

such that V: locally positive definite

$$\boxed{\begin{array}{l} a)\ V(0)=0 \\ b)\ V(x)>0 \text{ for all } x\in\mathcal{D}\setminus\{0\} \end{array}}$$

and $\frac{dV}{dt}=[\nabla V]^T\cdot f(x)\le 0$ for all $x\in\mathcal{D}$ [$\dot{V}$: locally negative semi-definite]

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then $\bar{x} = 0$ is stable in the sense of Lyapunov.