Lecture 18

03/22/23

Thm: Let $\mathcal{D}$ be an open connected subset of $\mathbb{R}^n$ that contains the eq. point $\bar{x}=0$ of $\dot{x}=f(x)$ $[f(0)=0]$.

1° If there is a continuously differentiable function

$$V:\mathcal{D}\longrightarrow\mathbb{R}$$

such that V: locally positive definite

a) $V(0)=0$

b) $V(x)>0$ for all $x\in\mathcal{D}\setminus\{0\}$

and c) $\frac{dV}{dt}=[\nabla V]^T\cdot f(x)\leq 0$ for all $x\in\mathcal{D}$ [$\dot{V}$: locally negative semi-definite]

1

then $\bar{x} = 0$ is stable in the sense
(this happens for pendulum) of Lyapunov.

2° If in addition to a) and b), we have

c2) $\frac{dV}{dt} = [\nabla V]^T \cdot f(x) \boxed{<} 0$, $\forall x \in D \setminus \{0\}$

then $\bar{x} = 0$ is $\boxed{\text{locally Asymptotically stable}}$

[local positive definiteness of V [a) & b)]]
local negative definiteness of $\dot{V}$ [c2]
along the solutions of $\dot{x} = f(x)$

3° If there is a cts diff'ble $V : \mathbb{R}^n \to \mathbb{R}$ with

a) $V(0) = 0$; b3) $V(x) > 0$, $\forall x \in \mathbb{R}^n \setminus \{0\}$

c3) $\dot{V} = [\nabla V]^T \cdot f(x) < 0$, $\forall x \in \mathbb{R}^n \setminus \{0\}$

d) $\lim_{\|x\| \to \infty} V(x) = +\infty$ [radial unboundedness]

then $\bar{x} = 0$ is globally asymptotically stable.

2

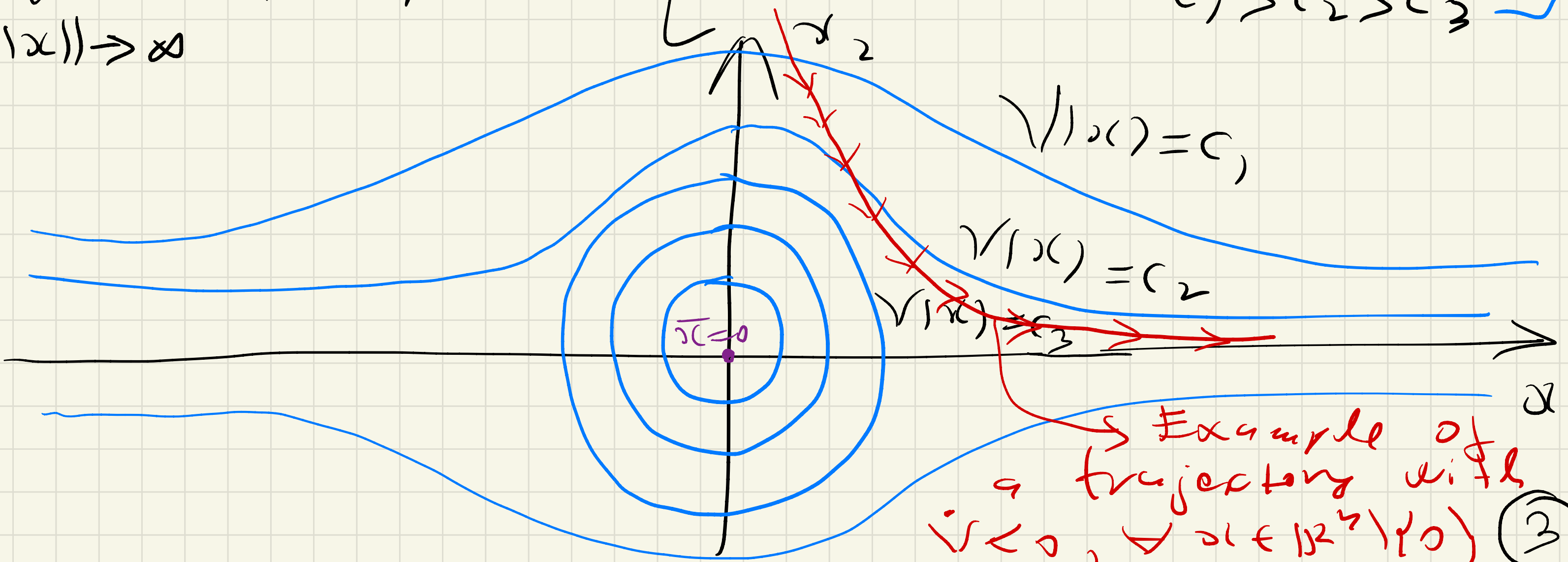
Significance of d)

$$V(x_1, x_2) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$

e.g. $x = \begin{bmatrix} x_1 \\ 5 \end{bmatrix}$; Even if $|x_1| \longrightarrow +\infty$

$$\lim_{\|x\| \to \infty} V(x) = 26$$ [Not radially unbounded]

$c_1 > c_2 > c_3$



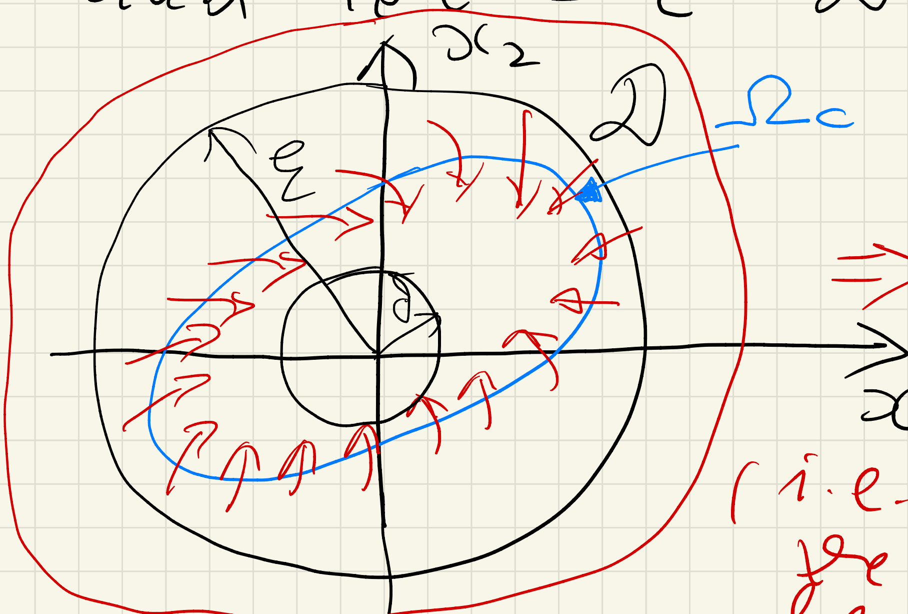
Proof of 1° & 2° (sketch)

1° Let V satisfy a), b), c)),

let $\Omega_c := \{x \in \mathcal{D};\ V(x) = c\}$,

constant

and let $\Omega_c \subset \mathcal{D}$



From c)) $\Rightarrow$

$\dot{V} \le 0,\ \forall x \in \mathcal{D}\setminus\{0\}$

$\Rightarrow$ each level set is positively invariant

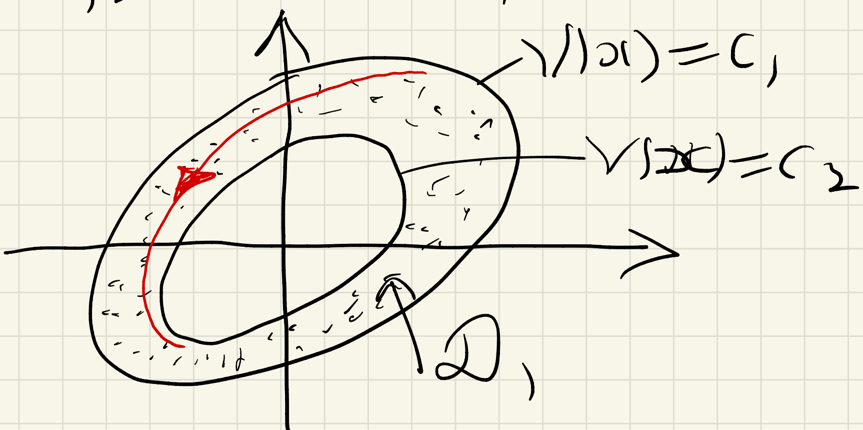
(i.e., if the start is the set, we'll never leave it) $\Rightarrow \bar{x} = 0$ is stable.

2° $\dot{V} < 0, \forall x \in D \setminus \{0\} \Rightarrow$

$\Rightarrow$ V is a decreasing function of time on D, which is bounded from below by $V(0) = 0$ $\Rightarrow$

$\lim_{t\to\infty} V(x(t))$ exists [i.e. $V(x(t))$ converges to e.g. $c_2 > 0$]

Q: Can $c_2 \neq 0$?



We'll prove that this is not possible!!!

Proof by contradiction

$\longrightarrow \quad \longleftarrow$

5

Assume that $\lim_{t\to\infty} V(|x(t)|) = c_2 > 0$

Let $\mathcal{D}_1 := \{x \in \mathcal{D};\ c_2 \le V(|x|) \le c_1\}$ s.t.

$$\max_{x\in\mathcal{D}_1} \dot{V} = -\gamma$$

Then $\frac{dV}{dt} \le -\gamma$ on $\mathcal{D}_1 \Longrightarrow$

$V(|x(t)|) \le V(|x(0)|) - \gamma \cdot t$ on $\mathcal{D}_1 \Longrightarrow$

There is $\bar{t} > 0$ s.t. $V(|x(\bar{t})|) < 0$

$\rightarrow\leftarrow \Longrightarrow c_2 = 0$.

Key challenge: How to construct Lyapunov functions???

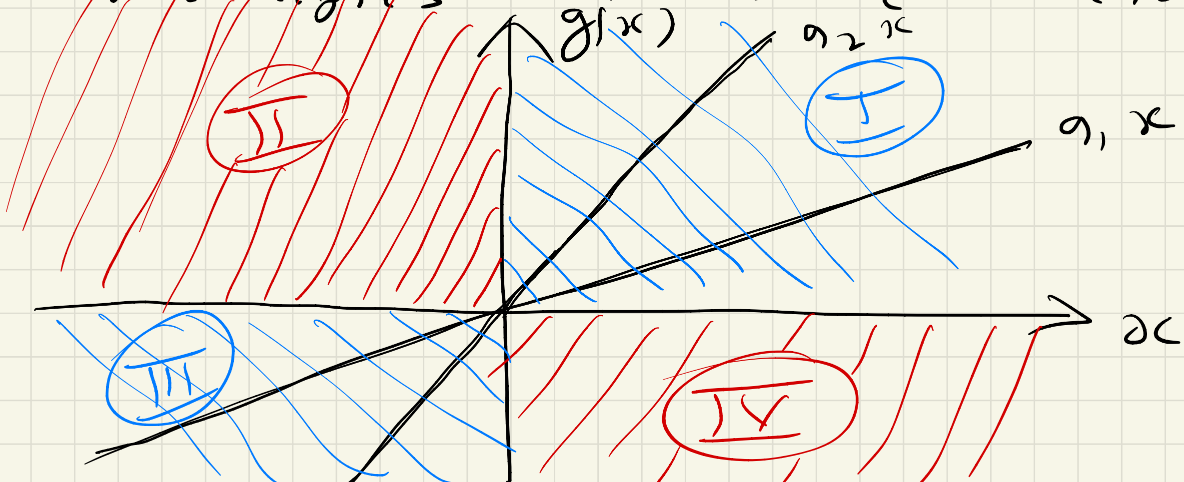
No UNIVERSAL Recipe!!!

Instead, we have guiding principles that allow us to exploit structure of nonlinear terms.

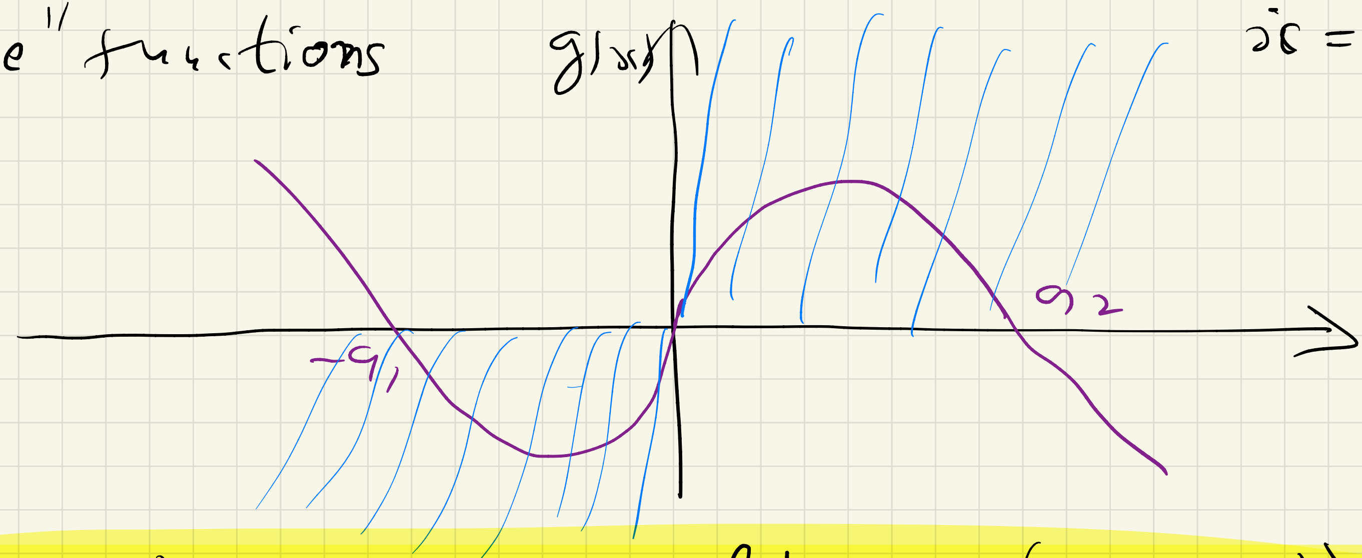
Ex 1: $\dot{x} = -g(x)$; $x(t) \in \mathbb{R}$: scalar problem

In LTI case, $g(x) = a \cdot x$ with $a > 0$ identifies all stable linear systems

$a_2 > a_1 > 0$



"Nice" functions $\qquad \dot{x} = -g(x)$



$x \cdot g(x) > 0$ for all $x \in (-a_1, a_2) \setminus \{0\}$

$g(0) = 0$

Lyapunov function candidate $V(x) = \frac{1}{2} x^2$

$\dot{V} = x \cdot \dot{x} = x \cdot (-g(x)) = -x \cdot g(x) < 0$

$\forall\ x \in (-a_1, a_2) \setminus \{0\} \Rightarrow$ LAS of $\bar{x} = 0$

8